Practice Exam 2

Math 527a – Principles of Analysis

(a) $\mathbb{N}$ is the set of natural numbers $1, 2, 3, \ldots$. $A \subseteq \mathbb{N}$ belongs to a collection $\mathcal{T}$ if $A = \emptyset$ or $\sup\{n : n \notin A\} < \infty$. Show that $\mathcal{T}$ defines a topology on $\mathbb{N}$. Is this topology Hausdorff?

Solution

$\emptyset \in \mathcal{T}$ by definition and $x \in \mathbb{N}^c \implies x \leq 0$, so it follows that $\mathbb{N} \in \mathcal{T}$.

If $U, V \in \mathcal{T}$ and $U \cap V \neq \emptyset$, it follows that $(U \cap V)^c = U^c \cup V^c$, so that $\sup\{n : n \notin U \cap V\} \leq \max(\sup\{n : n \notin U\}, \sup\{n : n \notin V\}) < \infty$, showing that $U \cap V \in \mathcal{T}$.

If $\{U_\alpha \,|\, \alpha \in \mathcal{A}\}$ is an arbitrary collection of open sets in $\mathcal{T}$ and the union $\left(\bigcup_{\alpha \in \mathcal{A}} U_\alpha\right) \neq \emptyset$, then there is at least one set, call it U_β such that $U_\beta \neq \emptyset$. Now looking at the compliment of the union we have

$$\left(\bigcup_{\alpha \in \mathcal{A}} U_\alpha\right)^c = \left[\bigcap_{\alpha \in \mathcal{A}, \alpha \neq \beta} U_\alpha^c\right] \cap U_\beta^c \subseteq U_\beta^c,$$

so that

$$\sup\left\{n : n \notin \left(\bigcup_{\alpha \in \mathcal{A}} U_\alpha\right)\right\} \leq \sup\left\{n : n \notin U_\beta\right\} < \infty$$

This shows that $\mathcal{T}$ is a topology.

Note also, if n_1 and n_2 are distinct elements of $\mathbb{N}$ and $n_1 \in U_1, n_n \in U_2$ where U_1 and U_2 are open, it follows that $U_1 \neq \emptyset$ so $\sup\{n : n \notin U_1\} = N_1 < \infty$ and likewise, $\sup\{n : n \notin U_2\} = N_2 < \infty$. If $n = \max(N_1, N_2) + 1$, it follows that $n \in U_1 \cap U_2$. Since U_1 and U_2 were arbitrary open sets containing n_1 and n_2 it follows that $\mathcal{T}$ is not Hausdorff.

(b) A topology on the space of smooth functions on $\mathbb{R}$ is defined by the sets

$$U(\phi, m.n.\epsilon) = \left\{\psi \text{ is a smooth function, } \left|\frac{d^k(\phi - \psi)}{dx^k}\right| < \epsilon, \forall |x| < n, 0 \leq k \leq m\right\}.$$

Define a map from smooth functions to $\mathbb{R}^2$ by $\phi \mapsto \left(\phi(0) - \phi(1), \int_0^1 [\phi'(x)]^2 dx\right)$. Show that this map is continuous.

Solution

Given a smooth function ϕ and $\eta > 0$. Let g denote the map in question, *i.e.*

$$g[\phi] = \left(\phi(0) - \phi(1), \int_0^1 [\phi'(x)]^2 dx\right).$$

Let $M = \max_{x\in[0,1]}(|\phi'(x)|)$. Since ϕ is smooth, $M < \infty$. We need to impose conditions on the function and it's first derivative over and interval $[0,1]$. Consequently, we look for neighborhoods of the form $U(\phi, 1, 1, \epsilon)$ with the goal of finding a suitable $\epsilon > 0$ such that

$$\psi \in U(\phi, 1, 1, \epsilon) \implies \|g[\psi] - g[\phi]\|_2 < \eta.$$

If $\epsilon < \eta/4$, it is clear that $\psi \in U(\phi, 1, 1, \epsilon) \implies |\phi(0) - \phi(1) - \psi(0) + \psi(1)| \le |\phi(0) - \psi(0)| + |\phi(1) - \psi(1)| < 2\epsilon = \eta/2$.

If $\epsilon < 1$, it follows that, for $0 \le x \le 1$

$$\begin{aligned}\left|[\phi'(x)]^2 - [\psi'(x)]^2\right| &= |\phi'(x) + \psi'(x)| \cdot |\phi'(x) - \psi'(x)| \\ &\le (2|\phi'(x)| + 1)|\phi'(x) - \psi'(x)| \\ &\le (2M + 1)|\phi'(x) - \psi'(x)|\end{aligned}$$

If $\epsilon < \eta/(4M+2)$ also, it is easy to see that

$$\begin{aligned}\left|\int_0^1 [\phi'(x)]^2 dx - \int_0^1 [\psi'(x)]^2 dx\right| &\le \int_0^1 |[\phi'(x)]^2 - [\psi'(x)]^2| dx \\ &\le (2M+1)\frac{\eta}{4M+2} = \frac{\eta}{2}\end{aligned}$$

Combining all of the results, we see that setting $\epsilon = \min(\eta, 1, \eta/(4M+2))$ guarantees that $\psi \in U(\phi, 1, 1, \epsilon) \implies \|g[\psi] - g[\phi]\|_2 < \eta$, thereby proving that the map is continuous.

(c) The notes define what it means for two bases to be equivalent. Write down the negation of this statement, *i.e.*, define what it means for two bases to *not be equivalent.* Using this or otherwise, show that if $\mathcal{T}_1$ and $\mathcal{T}_2$ are two first countable topologies on X, such that $x_n \to x$ in $\mathcal{T}_1$ if and only if it also converges in $\mathcal{T}_2$, then $\mathcal{T}_1 = \mathcal{T}_2$. (Hint: Identity map!)

Went over this in class.

(d) Prove or disprove: $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ is continuous. $\mathcal{T}$ is second countable. Then $\mathcal{S}$ is also second countable.

Counterexample

$X = Y = \ell^2(\mathbb{R}, \mathbb{N})$, $T = T_{metric}$, $S = T_{weak}$ and f is the identity map. Since $T_{weak} \subset T_{metric}$, it follows that f is continuous. However, T_{weak} is not first countable, and hece cannot be second countable, whereas l^2 is separable, so that T_{metric} is indeed second countable.

(e) If $\mathbf{x}^{(n)}$ converges weakly to $\mathbf{x}$ in $l^2(\mathbb{R}, \mathbb{N})$, show that, for each index i, $x_i^{(n)} \to x_i$. Using this or otherwise, show that (strongly) closed unit l^2 ball $\{\mathbf{x} \,|\, \|\mathbf{x}\|_2 \leq 1\}$ is also weakly closed. (Hint: Consider the functions $f_k(\mathbf{x}) = \sum_{i=1}^{k} x_i^2$. Note also that the (strongly) open unit ball $\{\mathbf{x} \,|\, \|\mathbf{x}\|_2 < 1\}$ is *not* weakly open!)

Let $\mathbf{e}^{(i)}$ be the ith vector in the "standard basis" of ℓ^2, that is the ith entry is one and the rest of the entries are zero. Since $\mathbf{x}^{(n)}$ converges weakly to $\mathbf{x}$, it follows that $x_i^{(n)} = \langle \mathbf{x}^{(n)}, \mathbf{e}^{(i)} \rangle \to \langle \mathbf{x}, \mathbf{e}^{(i)} \rangle = x_i$ proving the claim.

For each k, f_k is obtained as a finite sum of the compositions of continuous functions (with respect to weak convergence!) and is hence continuous. Consequently, if $\mathbf{x}^{(n)}$ converges weakly to $\mathbf{x}$, and $\|\mathbf{x}^{(n)}\|_2 \leq 1$, it follows that $f_k(\mathbf{x}^{(n)}) \leq 1$ and $f_k(\mathbf{x}^{(n)}) \to f_k(\mathbf{x})$, so that $f_k(\mathbf{x}) \leq 1$ for all k. Thus, $\|\mathbf{x}\|_2^2 = \lim_k f_k(\mathbf{x}) \leq 1$.

This proves that the strongly closed unit ball is also weakly closed.