Final Exam

Math 527a – Principles of Analysis

Problem 1 $a, b > 0$. Show that there is a constant $C < \infty$ such that

$$\int_0^\infty \frac{e^{-at}}{(1+bt^2)^{1/4}} dt \leq \frac{C}{a^{3/4} b^{1/8}}$$

(20 points)

Problem 2 A sequence $\mathbf{x}^{(n)}$ in $\mathbb{R}^{\mathbb{N}}$ *converges pointwise* to a sequence $\mathbf{x}$, if $x_i^{(n)} \to x_i$ for every index i.

(a) Prove or disprove : $\mathbf{x}^{(n)}$ is a sequence in $l^2(\mathbb{R}, \mathbb{N})$ that converges to $\mathbf{x} \in l^2$ pointwise. Then $\mathbf{x}^{(n)}$ converges to $\mathbf{x}$ in the l^2 norm. (15 points)

(b) Show that, if $\mathbf{x}^{(n)}$ converges to $\mathbf{x}$ in the weak topology on $l^2(\mathbb{R}, \mathbb{N})$ then it also converges to $\mathbf{x}$ pointwise. (15 points)

Problem 3 $\|\cdot\|_a$ and $\|\cdot\|_b$ are norms on a vector space V.

(a) Show that the function $p : V \to \mathbb{R}^+$ defined by $p(x) = \max(\|x\|_a, \|x\|_b)$ is a norm on V. (15 points)

(b) If $\|\cdot\|_{a^*}$ and $\|\cdot\|_{b^*}$ are the dual norms to $\|\cdot\|_a$ and $\|\cdot\|_b$ respectively, show that the norm p^* the dual to the norm p from part (a) satisfies

$$p^*(f) \leq \min(\|f\|_{a^*}, \|f\|_{b^*})$$

for all $f \in V^*$. (15 points)

Problem 4 $u(x)$ is a smooth solution of the Helmholtz equation

$$u''(x) - u(x) = f(x), \qquad u(0) = u(1) = 0$$

where $f(x)$ is a continuous function on $[0, 1]$. Show that there is a constant $C < \infty$ such that $\|u\|_\infty \leq C\|f\|_1$. (30 points)

Problem 5 $(X, \mathcal{T})$ is first countable. $A \subseteq X$. Show that $z \in \bar{A}$ the closure of A, if and only if there is a sequence $x_n \in A$ such that $x_n \to z$.

You are allowed to assume that at each $x \in X$, there is a countable collection of opens sets $V_1(x) \supseteq V_2(x) \supseteq V_3(x) \supseteq \cdots$ that is a local base for the topology. (30 points)

Problem 6 $X = C([0,1])$ and $\mathcal{T}$ is the metric topology on X induced by the L^2 norm.

(a) Let $U = \{f \in X | \quad \int_0^1 |f(x)|dx < 3\}$. Show that U is open in $(X, \mathcal{T})$. (15 points)

(a) Let $V = \{g \in X| \quad \|g(x)\|_\infty < 1\}$. Show that V is not open in $(X, \mathcal{T})$. (15 points)

Problem 7 Real valued sequences $x = \{x_k\}$ are indexed by $k = 1, 2, 3, \ldots$. Define an operator T by

$$(Tx)_k = \begin{cases} x_{k/2} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

(a) Show that T is a continuous linear map from $l^2(\mathbb{R}, \mathbb{N})$ into itself. (15 points)

(b) Show that, for all $x \in l^2(\mathbb{R}, \mathbb{N})$, the sequence $x^{(n)} = T^n(x)$ converges weakly to zero. (15 points)