Math 528A – Fall 09
Homework 2 : Due Sep. 30

2.1 The Minkowski Functional X is a linear space over $\mathbb{R}$.

(a) A point x is in the *interior* of a set $A \subseteq X$ if for all $z \in X$, there is an $\epsilon > 0$ such that $x + tz \in A$ if $|t| < \epsilon$. Show that we define a topology by declaring that a set O is open if all of its points are interior points.

(b) A convex set $K \subseteq X$ is said to be balanced if $x \in X \implies -x \in X$. Of course, this implies that the segment connecting x with $-x$ is in K so that $0 \in K$. A convex set K is absorbent if $\cup_{n \in \mathbb{N}} nK = X$. If K is a balanced, convex, absorbent set, show that 0 is in the interior of K.

The Minkowski functional or gauge of a convex absorbent set is given by

$$p_K(x) = \inf\{t \in [0, \infty) \,|\, x \in tK\}.$$

(c) A set K is bounded if for all $x \in K$, there exists a $t \in \mathbb{R}$ such that $tx \notin K$. If K is a bounded, convex, balanced, absorbent set, show that p_K is a norm on X.

2.2 Positive functionals Let S denote an arbitrary set and let $B(S)$ denote the set of bounded real valued functionals on S, *i.e.*

$$f \in B(S) \implies \|f\| \stackrel{\text{def}}{\equiv} \sup_{x \in S} |f(x)| < \infty.$$

(a) Show that $B(S)$ with the norm defined above is a complete, normed linear space.

There is a natural partial ordering on the elements of $B(S)$ given by $f_1 \leq f_2$ if $f_1(x) \leq f_2(x)$ for all $x \in S$. A function f satisfying $0 \leq f$ is said to be *non-negative.* A linear functional $\ell : B(S) \to \mathbb{R}$ is said to be positive if $\ell(f) \geq 0$ for all $f \geq 0$. This is a very useful notion. In particular, measures can be characterized as positive functionals on the space of continuous functions.

The goal of this problem is to prove a version of the Hahn-Banach theorem that allows us to extend positive functionals defined on a subspace to positive functionals on the whole space.

(b) Show that every positive linear functional is *monotone*, *i.e.*, $f_1 \leq f_2 \implies \ell(f_1) \leq \ell(f_2)$.

(c) $Y \subseteq B(S)$ is a subspace that contains a function larger than a constant function, *i.e*, W.L.O.G, there exists $y_0 \in Y$ such that $y_0(x) \geq 1$ for all $x \in S$. Also $\ell : Y \to \mathbb{R}$ is a positive functional. Show that, ℓ can be extended to a positive functional on all of $B(S)$.

2.3 Banach Limits Let $X = \ell^\infty$, the space of bounded real sequences. $c_0 \subseteq c \subseteq X$ are linear subspaces consisting of sequences that converge to 0 (c_0) and sequences which converge (c) respectively. $L : X \to X$ is the left shift operator, and $\lim : c \to \mathbb{R}$ is the limit. Both L and lim are linear operators. The (sequnce of) Césaro means are given by an operator $C : X \to X$ defined by

$$C(x_1, x_2, x_3, \ldots) = \left(x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \ldots\right)$$

(a) Show that c_0 and c are invariant spaces for the operators L and C.

In what follows, we will use the notation used in the proof of the Agnew-Morse theorem in class. Let $\mathcal{A}$ denote the semi-group of operators generated by L, *i.e.*, $\mathcal{A} = \{I, L, L^2, L^3, \ldots\}$ and $\mathcal{B}$ denote the semi-group of operators obtained by taking convex combinations of operators in $\mathcal{A}$. For $x \in X$, let

$$p(x) = \limsup_n x_n, \qquad g(x) = \inf_{B \in \mathcal{B}} p(Bx), \qquad h(x) = \max(g(x), g(-x)).$$

(b) Show that p, g and h are subadditive, positively homogeneous functions.

(c) If $x \in c_0$, show that $h(x) = 0$. Show that the converse is not true.

(d) Let $x = (1, -2, 0, 1, -2, 0, 1, -2, 0, ...)$. Find $g(x)$ and $h(x)$.

As defined in class, a *Banach Limit* LIM is an extension of the linear functional $\lim : c \to \mathbb{R}$ to a linear functional on all of X satisfying $\text{LIM}(x) \leq g(x)$. The existence of this extension is guaranteed by the Hahn-Banach theorem.

(e) Show that LIM is invariant under L.

(f) Show that the Banach limit can be chosen such that $\text{LIM}(x) = \lim(Cx)$ if the latter limit exists.

(g) A sequence is *almost convergent* if all Banach limits attain the same value on this sequence. Clearly, every convergent sequence is almost convergent. Show that there are *almost convergent* sequences which are not in c.

(h) Let ac_0 denote the linear subspace consisting of all the almost convergent sequences that belong to the nullspace of all the Banach Limits. Find a characterization of all these sequences in terms of the functionals g or h. Give an explicit example of a non-zero element in ac_0/c_0.

(i) Let $Y = X/c_0$. Show that p, g, h and C induce well defined operators on equivalence classes in Y.

(j) Show that L and C do not commute, but for all $x \in X$, $(LC - CL)x \in c_0$.

(k) Show that $g(Cx) = g(x)$ for all $x \in X$. Using this or otherwise, show that the Banach limit can be chosen so that it is, extends the lim operator, it is positive, and it is invariant under L and C.

2.4 Prove that if X is a Banach space and $\ell : X \to \mathbb{R}$ is a linear function. For $\alpha \in \mathbb{R}$, the hyperplane H_α is defined by $H = \{x | \ell(x) = \alpha\}$.

(a) If ℓ is not identically 0, show that there exists an $z \in X$ such that $\ell(z) = 1$ and if $x \in X$, then $x = y + \beta z$, where $y \in H_0$ and $\beta \in \mathbb{R}$.

(b) Show that every hyperplane $H_\alpha \in X$ is either closed or is dense in X.

2.5 X is a Banach space over $\mathbb{R}$. then X is naturally a linear space over $\mathbb{Q}$.

$F : X \to X$ is an *additive mapping* if $F(x + y) = F(x) + F(y)$ for all $x, y \in X$.

(a) If F is an additive map, show that F is a linear map over $\mathbb{Q}$.

(b) If F is continuous, show that F is a linear map over $\mathbb{R}$, and give an example to show that the hypothesis of continuity is necessary.

2.6 If X is a normed linear space over $\mathbb{R}$, the set of all bounded linear maps from X to $\mathbb{R}$ is called the *dual* of X and is denoted by X^*.

(a) Show that X^* is a complete normed linear space where the *induced norm* is defined by

$$\|\ell\|_{X^*} = \sup_{\|x\|_X = 1} |\ell(x)|$$

(b) Define the duality map from X to subsets of X^* by

$$x \mapsto F(x) = \{\ell \in X^* \mid \|\ell\|_{X^*} = \|x\|_X . \ell(x) = \|x\|_X^2\}.$$

Show that $F(x)$ is non-empty for all $x \in X$.

(c) X is strictly convex if

$$\|x + y\| = \|x\| + \|y\|, x \neq 0, y \neq 0 \quad \implies \exists a > 0 : x = ay.$$

If X is strictly convex, show that $F(x) \subseteq X^*$ reduces to a single point.

2.7 Show that the l^p norms on $\mathbb{R}^2$ are uniformly convex for $1 < p < \infty$. Optionally, show that this holds for the l^p norms on $\mathbb{R}^n$ for all n.