Math 528A – Fall 09
Homework 2 : Due Oct. 30

3.1 Quotient spaces X is a normed linear space and Y is a closed subspace of $\mathbb{R}$. For $[x] \in X/Y$, define $\|[x]\| = \inf_{z\in[x]} \|z\|$.

(a) Given $x \in X$, show that there exists a bounded linear functional $l \in X'$ such that $\|l\| = 1$ and $l(x) = \|[x]\|$. Show this implies that the null space of l is Y.

(b) Show that $\|[x]\|$ as defined above is a norm on X/Y.

(c) If X is a Banach space, show that X/Y is complete in the norm defined above.

As we defined in class, $Y^\perp$, the *annihilator* of Y is the set of all linear functionals $\ell \in X'$ that vanish on Y.

(d) Show that the dual $(X/Y)'$ is isometrically isomorphic to $Y^\perp$.

3.2 Closed linear span X is a normed linear space and $S \subseteq X$ is a set. The closed linear span of S is the smallest closed linear subspace containing S. Show that the closed linear span is the closure of the span of S.

3.3 Let H be a Hilbert space. Show that any two orthonormal bases in H have the same cardinality, *i.e* there is a bijection between the two bases.

3.4 Separable Hilbert spaces A Hilbert space is separable if it has a countable dense set.

(a) Show that a Hilbert space H is separable if and only if it has a countable orthonormal basis.

(b) Show that every separable Hilbert space (over $\mathbb{C}$) is isometrically isomorphic to $\ell^2(\mathbb{C})$, the space of square summable sequences with complex entries.

(c) Show that $L^2(\mathbb{R})$, the space of square integrable real values functions on $\mathbb{R}$ is a separable Hilbert space.

(d) If $\{x_j\}$ and $\{y_j\}$ are countable orthonormal bases for a Hilbert space H, show that the mapping $a \mapsto \sum(a, x_j)y_j$ gives an isometric isomorphism from H to itself, and further, every isometry of H is of this type.

3.5 Radon-Nikodym theorem (X, μ) is a measure space with $\mu(X) < \infty$. $\nu \ll \mu$ is another measure on X tha is *absolutely continuous* wit respect to μ. In class, we showed the existence of a measurable function y with $0 < y \leq 1$ (μ a.e.) such that for any $u \in L^2(d(\mu + \nu))$ we have

$$\int u(1-y)d\mu = \int uyd\nu$$

Use this to show that there exists a non-negative measurable function g such that for every measurable set $E \subseteq X$ we have

$$\nu(E) = \int_E g d\mu.$$

Show that this result also applies if the measure μ is σ-finite. (Hint: Monotone convergence theorem).

3.6 X is a Banach space and $Y \subseteq X$ is a closed linear subspace.

(a) Show that Y is also a Banach space and $Y' \subseteq X'$.

(b) If X is separable, show that Y is also separable.

(c) If X is reflexive, show that Y is also reflexive.

(d) Show that Y' is isometrically isomorphic to $X'/Y^{\perp}$, where $Y^{\perp}$ is the annihilator of Y. (Among other things, this implies the result in part (a)).

3.7 Attainment of boundary conditions in H_0^1.

3.8 The Obstacle problem