Math 528A – Fall 09
Homework 4 : Due Nov. 25

3.1 The Obstacle problem Let H_0^1 denote the Hilbert space obtained as the completion of the space of real valued smooth functions with support contained in $(-1,1)$ with respect to the inner product

$$(f,g) = \int_{-1}^{1} f'(x)g'(x)dx$$

$\Lambda(x)$ is a given smooth function with support in $(-1,1)$. We will refer to Λ as the *obstacle*.

(a) Show that, for each $y \in (-1,1)$, the evaluation map E_y given by $f \mapsto f(y)$ is a continuous map on H_0^1.

(b) We define the admissible set by $\mathcal{A} = \{f \in H_0^1 | f(x) \geq \Lambda(x)\ \forall x \in [-1,1]\}$. Show that $\mathcal{A}$ is closed and convex.

(c) The obstacle problem is to find a $u \in \mathcal{A}$ which minimizes $\|u\|^2 = (u,u)$. Show that there exists a unique solution to the obstacle problem.

(d) If u is a solution to the obstacle problem, and ϕ is a smooth, non-negative function with support in $(-1,1)$, show that $(u,\phi) \geq 0$.

3.2 X is a Banach space. A subset $S \subseteq X$ is weakly bounded if for all $\ell \in X'$, we have $\sup_{x\in S} |\ell(x)| < \infty$. Show that S is weakly bounded if and only if it is (strongly) bounded, *i.e.*, $\sup_{x\in S} \|x\| < \infty$.

3.3 Duality for ℓ^1

(a) Show that the dual of c_0 is ℓ^1.

(b) Show that the dual of ℓ^1 is ℓ^∞ but the dual of ℓ^∞ is not ℓ^1.

(c) Show that a sequence converges weakly in ℓ^1 if and only if it also converges strongly.

3.4 Finish the proof from class to show that a closed linear subspace of a reflexive Banach space is reflexive.

3.5 Support functions

X is a normed linear space. For any bounded subset $M \subset X$, we define the support function $S_M : X' \to \mathbb{R}$ by

$$S_M(\ell) = \sup_{y\in M} \ell(y).$$

Show that support functions have the following properties:

(a) For all $\ell, m \in X'$, we have $S_M(\ell + m) \leq S_M(\ell) + S_M(m)$.

(b) $a \geq 0 \implies S_M(a\ell) = aS_M(\ell)$.

(c) $M \subseteq N \implies S_M(\ell) \leq S_N(\ell)$.

(d) $S_{M+N} = S_M + S_N$.

(e) If $\bar{M}$ is the closure of M, then $S_{\bar{M}} = S_M$.

(f) The support function for the convex hull of M is the same as the support function of M.

(g) Using these properties (or otherwise), show that z is in the closed convex hull of a set M if an only if $\ell(z) \leq S_M(\ell)$.

3.6 Show that the unit ball in the dual of $C[-1, 1]$ is weak-* sequentially compact, but is not weakly sequentially compact. (You can do this without a complete explicit characterization of the dual space.)

3.7 $\theta_1, \theta_2, \ldots, \theta_n$ are n given points in $[0, 2\pi)$. Show that there is a continuous, 2π periodic function whose Fourier series diverges at each θ_i.