Homework # 1

August 31, 2005

"Gravity was invented by Isaac Walton. It is chiefly noticeable in autumn when apples are falling off trees" – *An anonymous student (courtesy the University of Utah Relativity group)*

1 Gronwall's inequality and separation of nearby solutions:

(a) $r : [0,t] \to \mathbb{R}$ is a continuous function such that

$$r(t) \leq a + b\int_0^t r(s)ds.$$

Show that $r(t) \leq ae^{bt}$.

(b) Obtain the appropriate modification to the conclusion above if r, a and b are continuous functions such that

$$r(t) \leq a(t) + \int_0^t b(s)r(s)ds.$$

Consider the system $\dot{x} = f(x)$ where $x \in \mathbb{R}$ and f is C^1. Assume that the first derivative f_x is bounded in absolute value by L on all of $\mathbb{R}$.

(c) If $\phi(x_0, t)$ and $\phi(y_0, t)$ are solutions of the differential equation with initial conditions x_0 and y_0 respectively, show that

$$|x(t) - y(t)| \leq |x_0 - y_0|e^{Lt}.$$

(d) Show that, every solution is bounded by

$$|x(t)| \leq \left(|x_0| + \frac{|f(0)|}{L}\right)e^{Lt}.$$

In particular, this shows that solutions do not become unbounded in finite time, and this ensures that the (maximal) solution exists for all time. (Hint: use the mean-value theorem to bound $|f(x) - f(0)|$)

2 Stability and Lyapunov functions

A *gradient flow* is a dynamical system $\dot{x} = -\nabla V(x)$, where $V : \mathbb{R}^n \to \mathbb{R}$ is a real valued C^2 function, that is referred to as the *potential* of the gradient flow.

(a) Show that $\frac{d}{dt}V \leq 0$ along orbits of the gradient flow.

(b) Show that the origin is an asymptotically stable fixed point for the two dimensional dynamical system

$$\begin{aligned}\dot{x} &= -2x + y + x^2 \\ \dot{y} &= x - 2y\end{aligned}$$

(c) identify the other fixed points for this system. Are they stable/asymptotically stable?

(d) Show that this dynamical system cannot have periodic orbits or homoclinic orbits.

3 Linear subspaces: Let $A \in \mathcal{M}_n(\mathbb{R})$ be a semisimple matrix and $x = x(t)$ a solution of

$$\dot{x} = A.x \tag{1}$$

$$x_0 = x(t_0) \tag{2}$$

Show that:

1. If $x_0 \in E^s$, then $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to-\infty} |x(t)| = \infty$
2. If $x_0 \in E^u$, then $\lim_{t\to\infty} |x(t)| = \infty$ and $\lim_{t\to-\infty} x(t) = 0$
3. If $x_0 \in E^c$, then $\exists\ m, M \in \mathbb{R}$ such that $\forall\ t\ \in \mathbb{R}$:

$$m \leq |x(t)| \leq M \tag{3}$$

4. Which of these properties hold if A is not semisimple? (prove or give a counter-example)

4 Liouville's theorem:

(a) Let $\Phi(t) = I + tA + O(t^2)$ be a matrix valued function of t, where I is the $n \times n$ identity matrix and A is a $N \times N$ square matrix. Show that

$$\frac{d}{dt}\det(\Phi(t))\bigg|_{t=0} = \text{trace}(A),$$

where det is the determinant, and the trace of A is the sum of it's diagonal entries.

(b) Let $\Phi(t)$ be a fundamental solution of

$$\dot{\Phi} = A(t).\Phi \tag{4}$$

$$\Phi(0) = I \tag{5}$$

Show that

$$\det \Phi(t) = \exp\left(\int_0^t \text{trace } A(s)ds\right). \tag{6}$$

(Hint: Observe that $\Phi(t+s) = (I + sA(t) + O(s^2))\Phi(t)$)

Note: This theorem is the the most simple form of the Liouville theorem which gives the contraction of volume in phase space. For a nonlinear system, the determinant of the fundamental solution of the variational equation is related to the *dissipativity* of the solutions. For Hamiltonian systems, the divergence of the vector field is the trace of the Jacobian of the vector field. In this case, using Liouville's theorem, it can be shown that the divergence vanishes, that is, the volume in phase space is conserved under the flow (Hamiltonians are therefore an example of *conservative systems* or *volume preserving systems*).

5 The variational equation and its adjoint: Consider the system $\dot{x} = f(x)$ where $x \in \mathbb{R}^n$ and $f(x)$ is a C^1 vector field. Let $\bar{x} = \bar{x}(t)$ be a particular solution. (i) Show that $\dot{\bar{x}}(t)$ is a solution of the variational equation around the solution $\bar{x}(t)$:

$$\dot{u} = Df(\bar{x}).u \tag{7}$$

This solution represents the tangent vector along the orbit. (ii) Use this property to build a fundamental solution for planar flows. Now, assume that the system has an autonomous first integrals $J = J(x)$. (iii) Show that $\partial_x J(\bar{x}(t))$ (that is, the gradient of the first integral evaluated on the particular solution) is a solution of the *adjoint variational equation*:

$$\dot{u} = -u.Df(\bar{x}) \tag{8}$$

where u is now a row vector. (iv) Show that with m independent first integrals one can build m independent solutions of the adjoint variational equations. (v) Finally, show that if Q is a fundamental solution of the variational equation then Q^{-1} is a solution of its adjoint (hence the name!).