

Homework # 3

October 8, 2005

Some of these problems involve numerical computation. If you need help with this, please come and see me.

Period 3 orbits and intermittency in the logistic map

The logistic map is given by $x_{n+1} = rx_n(1 - x_n) \equiv M(x_n, r)$.

(a) Show that the logistic map has a tangent bifurcation to a period 3 orbit at $r = 1 + \sqrt{8}$. What is the period three orbit that is born at this value of r ? (Hint: Consider the expression $M^3(x, r) - x$ and show that this polynomial has three double roots in addition to the simple roots at $x = 0$ and $x = 1 - 1/r$. A symbolic computation program like mathematica or maple will help here)

(b) Following the derivation in class, find the (approximate) average length of the laminar phase at a parameter value $r = 1 + \sqrt{8} - \epsilon$.

(c) Verify the result from (b) by direct numerical computation using a range of ϵ values (say from $\epsilon = 10^{-6}$ to $\epsilon = 10^{-1}$. Also, pick a value of c , say 0.1, and define the laminar phase by a stretch of time where every third iterate of the map is within c of the middle value of the periodic orbit from (a).)

(d) If l is the length of a laminar phase, the scaled length \tilde{l} is $l/\sqrt{\epsilon}$. On the same graph, plot histograms of the relative frequencies of the scaled lengths \tilde{l} for different values of ϵ . What do you observe? Can you explain this observation analytically?

Self-similarity in the Logistic map and the Feigenbaum attractor

A crude value for the critical parameter value r_∞ is 3.59.

(a) Numerically determine the values $R_k < r_\infty$ at which the logistic map has superstable orbits of period 2^k for $k = 1, 2, \dots, 12$. (Hint: one possible approach is to use the matlab function `fzero` and use the crude value for r_∞ as an upper bound for the search region.)

(b) Determine the distance d_k between $1/2$ and the closest point on the 2^k superstable orbit which occurs at $r = R_k$.

(c) From the results of parts (a) and (b), calculate the ratios d_i/d_{i+1} and $(R_i -$

$R_{i-1})/(R_{i+1} - R_i)$. What do these calculations suggest? Estimate the Feigenbaum constants α and δ from this. Also obtain a better estimate for r_∞ .

In what follows, use the values of α and r_∞ obtained in part (c).

(d) Numerically calculate an orbit with $x_0 = 1/2$ and $r = r_\infty$ for the logistic map. Plot $x_n - 1/2$ along this orbit, as well as rescalings obtained by repeatedly flipping and then shrinking the y axis by a factor $\alpha \approx 2.5$ and shrinking the x -axis (time index) by 2. What do you observe? (Note: Make sure your rescaled x 's and y 's are in the same range in all the graphs)

(e) Argue that the results from (d) suggest that, with $r = r_\infty$,

$$M^{2k} \left(\frac{1}{2}, r_\infty \right) - \frac{1}{2} \approx \frac{-1}{\alpha} \left[M^k \left(\frac{1}{2}, r_\infty \right) - \frac{1}{2} \right]$$

and this approximation gets better and better as $k \rightarrow \infty$.

(f) What does part (d) suggest about the behavior of orbits whose initial condition are not $x_0 = 1/2$. Can you write down a relation similar to the approximate relation in (e)?

(g) Consider the sequence of maps defined by

$$g_i(y) = (-\alpha)^i \left(M^{2^i} \left[\frac{1}{2} + \frac{y}{(-\alpha)^i}, r_\infty \right] - \frac{1}{2} \right)$$

Plot these functions for $y \in [-1, 1]$. What do you observe?

(h) Assuming that the sequence of functions g_i in part(g) has a limit, obtain an equation characterizing the limit. Does this limit depend on M ?

Universality and the period doubling sequence for unimodal maps

The cubic map is given by $x_{n+1} = rx_n(1 - x_n^2)$.

(a) Obtain a bifurcation diagram for the cubic map, and estimate a crude value of r_∞ from this bifurcation diagram.

(b) Repeat parts (a)-(d) of the previous problem for the cubic map. what do you obtain for α, δ and r_∞ ? Note that, for part (d), rather than start with $x_0 = 1/2$, you now have to start in the vicinity of the maximum of the cubic map.

(c) Following the idea in part (g), define an appropriate sequence of maps g_i for the cubic map. Does this sequence look like it is converging? Plot the sequence g_i for the logistic map and also for this map on the same plot. what do you observe?