Homework # 4

December 1, 2005

Some of these problems involve numerical/symbolic computation. If you need help with this, please come and see me.

Duffing's Equation

Duffing's equation is

$$\ddot{x} + \delta\dot{x} - x + x^3 = \gamma\cos(\omega t).$$

(a) Show that all solutions are bounded if $\gamma = 0, \delta \geq 0$.

(b) Show that the orbits are bounded even for $\gamma > 0, \delta \geq 0$.

(c) Using an ODE solver, find the saddle periodic orbit and it's stable and unstable manifolds for $\delta = 0.25, \omega = 1$ and (i) $\gamma = 0$ (ii) $\gamma = 0.1$, (iii) $\gamma = 0.2$ and (iv) $\gamma = 0.3$.

Bifurcations and transversality

For each of the following functions $f(x,\mu)$, draw the bufurcation diagram for $\dot{x} = f(x,\mu)$ near $\mu = 0$. Also, identify if the bifurcation type is "generic" (stable under perturbation), and if it is not generic, embed it into a system with more parameters, so that it now occurs in a persistent manner.

(a) $f(x,\mu) = \mu - x^2$. (Saddle-Node bifurcation).

(b) $f(x,\mu) = \mu x - x^2$. (Transcritical bifurcation).

(c) $f(x,\mu) = \mu x - x^3$. (Pitchfork bifurcation).

(d) $f(x,\mu) = \mu^2 x - x^3$.

(e) $f(x,\mu) = \mu^2 x + x^3$.

(f) $f(x,\mu) = \mu^2\alpha x + 2\mu x^3 - x^5$, for $\alpha = -1, 0, 1$.

Center manifolds and stability

Using a power series expansion, find an (approximate) center manifold W^c and the reduced dynamics on W^c for the system

$$\begin{aligned}\dot{x} &= \alpha x^2 - y^2 \\ \dot{y} &= -y + x^2 + xy\end{aligned}$$

Circle maps and phase locking

The circle map is given by

$$\theta_{n+1} = f(\theta_n) \equiv \theta_n + \Omega - \frac{K}{2\pi}\sin(2\pi\theta_n)$$

The rotation number of an orbit is defined by

$$R(\theta_0) = \lim_{n\to\infty} \frac{f^n(\theta_0) - \theta_0}{n}$$

(a) Show that, for $0 \leq K \leq 1$ the map f is monotone, and does not undergo period doubling bifurcations as either K or Ω is varied.

(b) For $0 \leq K \leq 1$, show that the rotation number is independent of the initial condition θ_0 and only depend on K and Ω.

(c) Show that the rotation number of a periodic orbit is rational, and conversely, if the rotation number $R(\theta_0)$ is a rational number, then the orbit of θ_n converges to a periodic orbit.

(d) By numerically following orbits of the circle map identify the Arnold tongues corresponding to rotation numbers $0, 1, 1/2, 1/3$ and $2/3$.

As we argued in class, keeping $K > 0$ fixed and varying Ω results in periodic orbits being created and then destroyed by saddle node bifurcations.

For a periodic orbit with rotation number p/q, we have

$$f^q(\theta) = p + \theta$$

In addition at the parameter values where the orbits are created/destroyed, we have

$$\frac{\partial}{\partial\theta} f^q(\theta) = 1.$$

(e) Analytically determine the boundaries of the Arnold tongue corresponding to $R = 0$ and $R = 1$.

The analysis of the other wedges close to $K = 0$ can be carried out as follows, where we identify the leading behavior as $K \to 0$.

As $K \to 0$ along the boundaries of the (p, q) Arnold tongue, we have $\Omega \to p/q$ and $\theta \to \alpha^{\pm}$ along the right and the left boundaries respectively.

(f) For $p/q = 1/2$, use the expansions $\theta = \alpha^{\pm} + o(1)$ and $\Omega = 1/2 + c_1^{\pm} K + c_2^{\pm} K^2 + O(K^3)$ in the two conditions above to show that the boundaries are given by $c_1^{\pm} = 0, c_2^{\pm} = \pm\frac{1}{8\pi}$. What are $\alpha^{\pm}$.

(g) The results from the previous part suggest that the natural scalings for the (1,2) wedge are $\theta = O(1), \Omega - 1/2 = O(K^2)$. We can numerically verify this scaling as follows:

Let $\phi(x, \eta, K)$ be defined by

$$\phi(x, \eta, K) = \frac{f(f(x)) - 1 - x}{K^2}$$

where $\Omega = 1/2 + \eta K^2/(8\pi)$. Plot the function $\phi(x, \eta, K)$ as a function of x for different choices of η and K. Do the curves converge as $K \to 0$ with fixed η? What do you observe about the curves for $\eta = 1, 0, -1$? Finally compare the behavior for $K = 1$ with the behavior for small K, and explain what this means for the validity of the scaling analysis when K is not extremely small.