Homework # 1

January 23, 2006

Some of these problems involve numerical/symbolic computation. If you need help with this, please come and see me.

Stability

A fixed point $x_0 \in X$ of a topological dynamical system $f : X \to X$ is *stable* if for every open set U containing x_0, there exists another open set V such that for every $y \in V$, $f^k(y) \in U$ for $k = 0, 1, 2, 3, \ldots$.

(a) How would you extend this idea to define the stability of a periodic orbit of $f : X \to X$?

(b) Assuming that open neighborhoods are given by a metric d on X, rewrite the definition of stability of a fixed point in terms of the metric.

(c) If $\sigma : \Omega_N \to \Omega_N$ represents the full shift on N symbols, does σ have stable fixed points or periodic orbits? Justify your answer with a proof or an example.

(d) If X contains two disjoint open sets U and V, and $f : X \to X$ is topologically transitive, show that f does not have stable fixed points or periodic orbits. (Hint: Part (c) is a special case of this result.)

Topological Markov chains

α and β are two symbols. Ω_A consists of all infinite sequences of these symbols such that every set of three consecutive symbols are one of the following admissible subsequences:

$$\alpha\alpha\beta, \alpha\beta\beta, \beta\beta\alpha, \beta\alpha\beta \quad \text{and} \quad \beta\alpha\alpha.$$

(a) Argue that this is a 2 step Markov chain, and draw the corresponding Markov graph using states corresponding to pairs of symbols.

(b) Is $\Omega_A = \Omega_2$, the set of all sequences with two symbols?

(c) If $\sigma : \Omega_A \to \Omega_A$ is the shift map, show that σ is topologically mixing.

(d) Compute $P_n(\sigma)$, the number of periodic orbits of period n.

Non-negative matrices

A matrix A is *non-negative*, or $A \geq 0$ is every entry of A is non-negative, and likewise, $A > 0$ (A is positive) if every entry of A is positive. Also, $A \geq B$ is $A - B \geq 0$ and $A > B$ if $A - B > 0$.

(a) Show that, if $A \geq B \geq 0$, then $A^k \geq B^k \geq 0$ for all $k = 1, 2, 3, \ldots$.

(b) Show by explicit example that, we can have $A \neq B$, $A \geq B \geq 0$ but $A^2 = B^2$.

To each matrix non-negative matrix A, we can associate a $0 - 1$ matrix A' by $A'_{ij} = 1$ is $A_{ij} > 0$ and $A'_{ij} = 0$ otherwise.

(c) Show that $A^k > 0$ for some k if and only if A' is transitive.

(d) Given a $m \times m$ matrix $A \geq 0$, show that, there is an integer $N(m)$, that depends on m, but is independent of A such that $A^k > 0$ for some k if and only if $A^{N(m)} > 0$.

This result has an important implication. To check whether $A^k > 0$ for some k seems like checking an infinite number of conditions. However, this result implies that we can figure out whether $A^k > 0$ for some k (or if A' is transitive) in a finite number of steps.

(e) If $A' \geq I$ (the identity matrix), then A' is transitive if and only if $(A')^{m-1} > 0$. (m is the size of the matrix)

(f) If an $m \times m$ A' is transitive, show that for each $i = 1, 2, 3, \ldots, m$, there is an integer $k_i \leq m$ such that the entry $(A')^{k_i}_{ii} > 0$.

(g) Combining (e) and (f), can you find a bound for $N(m)$ in part (d)?