Homework # 2

February 16, 2006

Some of these problems involve numerical/symbolic computation. If you need help with this, please come and see me.

Invertible and non-invertible systems

A "standard" way to make an invertible map out of a non-invertible 1-d map $x_{n+1} = f(x_n)$ is to increase the dimensionality of the system and consider the 2-d map

$$\begin{aligned} x_{n+1} &= f(x_n) + b y_n \\ y_{n+1} &= x_n \end{aligned} \tag{1}$$

(a) Show that this 2-d map from $\mathbb{R}^2 \to \mathbb{R}^2$ is indeed invertible.

(b) Show that if $|b| < 1$, the map in (1) contracts areas in $\mathbb{R}^2$.

(c) We are given that $f(x) = 4xe^{-x^2}$ and $|b| < 1$. Show that there is a bounded set $K \subset \mathbb{R}^2$ such that every orbit of the map in (1) eventually enters the set K.

(d*) Numerically compute the long time-behavior of orbits for $b = 1/2$. Does the long-time dynamics settle on to a set that can be described by the graph of a function $y = g(x)$? Can you justify/prove this analytically?

(d) If the map f is a continuous, non-invertible map on the unit-circle, can you construct a continuous, invertible map on the Cylinder $S^1 \times \mathbb{R}$ using this idea? How about a continuous invertible map on the Torus? What values of b are allowed in the two cases.

(e) Show that there is no invertible C^1 map from the torus into itself that contracts areas everywhere.

(f) Can you find an example of a non-constant map $\phi : T^2 \to T^2$ from the Torus into itself which does contract areas everywhere? If the lift of ϕ onto $\mathbb{R}^2$ is $\Phi = M + \xi$, where M is a matrix with integer entries and ξ is doubly periodic, show that $\det(M) = 0$.

Hyperbolic toral automorphisms

M is a matrix with integer entries and non-zero determinant.

(a) Show that the map on the Torus induced by the "affine" map $(x, y) \mapsto (x, y)M^T + (x_0, y_0)$ is k to 1, where $k = |Det(M)|$.

We will henceforth denote the induced map on the Torus by f_M and assume that for some integers $b, k > 0$

$$M = \begin{pmatrix} b^2 + k & b \\ b & 1 \end{pmatrix}.$$

(b) Show that f_M has a fixed point unless M is the identity matrix.

(c) Show that, if ξ is a doubly periodic C^1, then the (perturbed) map f_M^ϵ induced by $(x, y) \mapsto (x, y)M^T + (x_0, y_0) + \epsilon\xi(x, y)$ is also k to 1 for sufficiently small ϵ.

(d) Prove or give a counter-example: For sufficiently small ϵ $P_n(f_M) = P_n(f_M^\epsilon)$. (Hint: Structural stability! If you can come up with a direct proof, this will give an alternate way to construct the conjugating homeomorphism)

(e*) Assume that $k = 1$. In class, we considered the Markov partition of the torus for the case $x_0 = y_0 = 0$, $b = 1$. Can you describe how to construct the partitions, and also the induced Markov graph for $b = 2$? For arbitrary b?

Contraction Mapping and Hyperbolic systems

(a) Show that the map

$$\begin{aligned} x_{n+1} &= 2x_n \\ y_{n+1} &= by_n + 2\cos^2(\pi x_n) - \sin(2\pi x_n) \end{aligned}$$

induces a smooth map on the cylinder, for all values of b.

(b) Show that the map has an invariant set that is given as the graph of a continuous function $y = g(x)$, as long as $|b| \neq 1$.

(c) Numerically compute (an approximation to) the function g if $b = 1/2$ and $b = 2$. Is the function C^1? Can you prove your claim? (Hint: Look at the Fourier coefficients of g)