Homework # 3

April 6, 2006

Some of these problems involve numerical/symbolic computation. If you need help with this, please come and see me.

Shadowing Property

In the previous homework, we considered the map

$$x_{n+1} = 2x_n \bmod 1$$
$$y_{n+1} = by_n + 2\cos^2(\pi x_n) - \sin(2\pi x_n)$$

defined on the cylinder $S^1 \times \mathbb{R}$.

We saw that a direct numerical simulation of this map is not possible, because roundoff errors will cause every orbit to converge to $x_n = 0$. We will figure out a way to get around this problem.

Consider the map $z_{n+1} = (2z_n + r_n) \bmod 1$, where r_n is "noise" with the property that z_{n+1} and $2z_n$ agree to the first m binary digits, and the rest of the digits of z_{n+1} are random.

(a) Show that, given an "orbit" $\{z_k\}$ for the "random map", there is a "true" orbit $\{\tilde{x}_k\}$ with $\tilde{x}_{n+1} = 2\tilde{x}_n \bmod 1$, such that $|\tilde{x}_k - z_k| \leq 2^{-m+1}$ for all k. This is the (infinite-time) shadowing property. (Note: This is not standard terminology)

(b) Show that the family of measures

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{z_i}$$

converges to Lebesgue measure on $[0, 1]$.

(c) Show that, the orbits for the "noisy" 2-d map

$$z_{n+1} = (2z_n + r_n) \bmod 1$$
$$y_{n+1} = by_n + 2\cos^2(\pi z_n) - \sin(2\pi z_n)$$

also have the (infinite-time) shadowing property. What is the maximum difference in y between the true and the noisy orbit?

(d) Describe an algorithm for implementing this idea on a computer. Also, plot the asymptotic behavior of an orbit of the "noisy" map for $m = 10, 20$ and 30 (correpsonding roughly to deviations of size $10^{-3}, 10^{-6}$ and 10^{-9}).

(e) Plot the behavior of an orbit as in the previous part, where r_n is uniformly distributed between $\pm 2^{-30}$. Do you see any difference from the plot in (d)?

The Gauss Endomorphism

For $x \in (0,1)$, the Gauss Endomorphism is defined by

$$\varphi(x) = \left\{\frac{1}{x}\right\}$$

where $\{.\}$ denotes the fractional part.

(a) Show that φ defines a dynamical system on $(0,1)/\mathbb{Q}$, the set of all the irrationals on $(0,1)$.

(b) Show that the measure μ given by the density $\rho(x) = (\ln(2)(1+x))$,

$$\mu(A) = \frac{1}{\ln(2)} \int_A \frac{dx}{1+x}$$

is an invariant measure for φ, by (a) verifying the invariance for sets of the form $(0,\alpha)$ and (b) verifying that ρ satisfies the Frobenius-Peron equation.

Although we can check that we have an invariant measure, the existence of such a measure cannot be deduced directly since we do not have a continuous map form a compact space into itself. In particular, it is not clear that a small perturbation of the Gauss endomorphism will also have an invariant measure with a continuous density. So, we will try to prove the existence of an invariant measure, without knowing the exact answer.

The map φ is not continuous. Define the map $\phi_n : [0,1] \to [0,1]$ by

$$\phi_n(x) = \begin{cases} \varphi(x) & x \in [1/n, 1] \\ 1 - nx & x \in [0, 1/n] \end{cases}$$

(c) Show that the maps ϕ_n give continuous maps on the unit circle with the usual identification $z = \exp(2\pi i x)$.

(d) Show that the sequence of maps ϕ_n converges to φ a.e (with respect to Lebesgue measure), but the sequence does not converge uniformly. (Hint: What is the degree of the map ϕ_n?)

(e) What is the topological entropy of ϕ_n? Show that $h_d(\varphi) \geq h_d(\phi_n)$, where h_d denotes the topological entropy. What does this imply about $h_d(\varphi)$?

(f) If μ_n is an invariant probability measure for ϕ_n, show that $\{\mu_n\}$ has a subsequence that converges weak-$*$ to a measure μ that is invariant for φ.

(g$*$) Consider the "stochastic" map given by

$$\zeta_n(x) = \begin{cases} \varphi(x) & x \in [1/n, 1] \\ \xi & x \in [0, 1/n] \end{cases}$$

where ξ is a random variable uniformly distributed in $[0,1]$. Derive the equation that determines the invariant density $\rho_n(x)$ for ζ_n. Show that each ζ_n has a unique invariant density ρ_n, these densities are uniformly bounded, and they have a convergent subsequence that converges to an invariant density for φ.