Math 527B – Spring 09
Homework 12 : Don't need to turn this in.

12.1 Prove Fatou's lemma from the monotone convergence theorem.

12.2 Problem 2.5.20, Pg. II-366.

12.3 Show with proof that $\lim_{N\to\infty} \sum_{k=0}^{N} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$.

12.4 Show that, for $\alpha \in (0,1]$, there is a constant C_α such that $\int_0^\infty \frac{e^{-\lambda t}}{\sqrt{1+t^2}} \leq \frac{C_\alpha}{\lambda^\alpha}$ for all $\lambda > 0$, but the analogous inequality for $\alpha = 0$ is false. (Hint: Hölder's inequality for one part, monotone convergence theorem for the other)

12.5 Problems 2.5.32 and 2.5.33, Pg. II-370.

12.6 **Optional** Look at all the problems in Sec. II.2.5, including the miscellaneous exercises on Pgs. II-370 – 372, and bring your questions to me or Ben.

12.7 Compute the following limit or show that it doesn't exist

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(1 + \frac{k}{n^2}\right)^n$$

Carefully justify all your steps, making sure you state the names of any convergence theorems that you use.

12.8 Show, with justification, that

$$\int_0^\infty \frac{x}{e^x - 1}\, dx = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Note that

$$\int_0^\infty x e^{-nx} dx = \frac{1}{n^2}$$

12.9 For what values(s) of α is the following limit finite and non-zero? Compute, with justification, the value of the limit for this choice of α.

$$\lim_{n\to\infty} n^\alpha \int_0^n \frac{e^{-n(x-n+\sin(n))^2}}{1+x^2} dx$$

12.10 Problem 2.6.28, Pg. II-383.

12.11 Problem 2.6.32, Pg. II-383.

12.12 (a) Show that there exists a sequence of measurable non-negative functions f_n such that $\int f_n = 1/n$ but f_n does not converge to zero at any point in $\mathbb{R}$.

(b) $f \in L^1$ is a non-negative function. Define a sequence of measurable functions by $g_n(x) = f(n^2 x)$. Show that g_n converges to zero pointwise (a.e.).

(c) h_n is a sequence of measurable, nonegative, functions such that $\int h_n \leq 1/n^2$. Show that the sequence h_n converges to zero pointwise a.e.