Math 527B – Spring 09
Homework 3 : Due Feb. 13

3.1 Expressing distributions in various coordinate systems

In class, we saw that, if $\psi(x, y)$ is a smooth function expressed in cartesian coordinates, and $\phi(r, \theta) = \psi(r\cos(\theta), r\sin(\theta))$ is the same function expressed in polar coordinates,

$$\iint \delta(x^2 + y^2 - \lambda^2)\psi(x, y)dxdy = \frac{1}{2}\int_0^{2\pi} \phi(\lambda, \theta)d\theta$$

I will refer to this identity as expressing the action of the distribution $\delta(x^2+y^2-\lambda^2)$ in polar coordinates.

(a) Show that

$$\iint \delta(r^2 - \lambda^2)\phi(r, \theta)rdrd\theta = \frac{1}{2}\int_0^{2\pi} \phi(\lambda, \theta)d\theta$$

and using this, show that $\delta(r^2 - \lambda^2)$ has the same action on smooth functions as the distribution $\delta(x^2 + y^2 - \lambda^2)$ except in a different coordinate system. Generalize from this example to give a rule for change of coordinates for distributions.

(b) Express the action of $\delta(x^2 + y^2 - \lambda^2)$ on a smooth function $\psi(x, y)$ in cartesian coordinates.

3.2 Distributions in 3 dimensions

Let (x, y, z) denote cartesian coordinates, (r, θ, ϕ) denote spherical polar coordinates and (ρ, z, ϕ) denote cylindrical polar coordinates. Determine the action of the following distributions on compactly supported smooth test functions in an appropriate coordinate system.

(a) For example, show that

$$\delta(\|x\| - a)(\psi) = a^2 \int_0^{2\pi}\int_0^{\pi} \psi(a, \theta, \phi)\sin\theta \; d\theta\, d\phi$$

Give similar results for the following:

(b)

$$\frac{1}{1+x^2}\delta(x^2 - y^2 - z^2)$$

(c)

$$\delta(\|x\|^2 - a^2) = \delta(r^2 - a^2)$$

(d)

$$\delta(\sin(z))\delta(\rho^3 - \rho)$$

(e) $e^{-z^2}\delta(\rho - z)$. For this part, express the answer in all the coordinate systems, and from this, express the distribution in cartesian and spherical coordinates.

(f) $\delta(G(x,y,z)$ where $G : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function such that $G(x_0,y_0,z_0) = 0 \implies \nabla G(x_0,y_0,z_0) \neq 0$.

3.3 Construction of smooth functions

(a) $a < b$ are given real numbers. Construct a smooth function ϕ such that $\text{supp}(\phi) \subseteq [a,b]$ and $\phi \geq 0$. (There is a construction in the book, and we talked about this in class. Fill in all the details.)

(b) $a < b$ are given real numbers. Construct a smooth non-negative function ψ supported on $[a,b]$ such that $\int_a^b \psi(x)dx = 1$.

(c) A *smooth transition layer* is a C^∞ function φ such that

$$\varphi(x) = \begin{cases} \alpha & x \leq a \\ \in [\alpha,\beta] & a \leq x \leq b \\ \beta & x \geq b \end{cases}$$

Construct such a function. (See also Problem 4.2.34, Pg. I-207)

(d) Given real numbers $a < b$ and $\epsilon > 0$, construct a nonegative smooth function ζ such that

$$\zeta(x) = \begin{cases} 1 & a \leq x \leq b \\ \in [0,1] & x \in [a-\epsilon,a] \cup [b,b+\epsilon] \\ 0 & \text{otherwise} \end{cases}$$

(See also Problem 4.2.32, Pg. I-207)

(e) There are occasions where one needs an explicit formula for a smooth transition layer. Using the fact that the hyperbolic tangent tanh is a smooth function, give an explicit formula for a smooth function η satisfying

$$\eta(x) = \begin{cases} -1 & x \leq -1 \\ \in [-1,1] & -1 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases}$$

Plot $\eta(x)$ and $\eta'(x)$. Can you find a function η such that $\eta' \geq 0$ for all x and further, η' has a maximum at $x = 0$ and is monotone for $x < 0$ and $x > 0$.

(f) Problem 4.2.35, Pg. I-207 in the book.

3.4 Let ψ be the compactly supported smooth function constructed in part(b) above.

(a) $f \in C^\infty(\mathbb{R})$. Show that the following limit exists and evaluate the limit:

$$\lim_{j \to \infty} \int_{\mathbb{R}} f(x)\,[j\psi(jx)]\,dx$$

Does the limit depend on ψ?

(b) Show by explicit example, that there are smooth functions f for which (whom?) the following limit does not exist:

$$\lim_{j\to\infty} \int_{\mathbb{R}} f(x) \left[j\psi(jx^2) \right] dx$$

(c) Following the idea of regularizing by "canceling" the singular contribution near zero to regularize this integral, we consider the limit:

$$\lim_{j\to\infty} \int_{\mathbb{R}} f(x) \left[j\, \mathrm{sgn(x)}\psi(\mathrm{jx}^2) \right] dx$$

where sgn(x) $= \pm 1$ if $x > 0$ and $x < 0$ respectively and sgn(0) $= 0$. Show that the limit exists, and defines a linear functional.

(d) If $z(x)$ is a differentiable function with *non-degenerate zeros*, that is, $z(x_0) = 0 \implies z'(x_0) \neq 0$. $f \in C_0^\infty(\mathbb{R})$. Show that the following limit exists and evaluate the limit:

$$\lim_{j\to\infty} \int_{\mathbb{R}} f(x) \left[j\psi(jz(x)) \right] dx$$

Does the limit depend on ψ?

(e) In keeping with the "philosophy" that we discussed in class, the limit in part (c) is an appropriate definition for $\delta(z(x))$. If $z(x)$ is differentiable with non-degenerate zeros, what is $\delta(z(x))(\phi)$ for a test function ϕ? use this formula to solve Problem 4.1.71, Pg. I-195.