

Math 527B – Spring 09
Homework 4 : Due Feb. 25

4.1 Show that the family of semi-norms

$$\zeta_{m,k}(\varphi) = \sup_{x \in \mathbb{R}^n} \max_{0 \leq |\alpha| \leq k} (1 + \|x\|^m) |D^\alpha(\varphi(x))|$$

generates the usual topology on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. (These norms are easier to use for problems, since they also control the behavior of the function near 0).

4.2 Problems 4.2.62 and 4.2.63, Pg. I-216 in the book.

4.3 Problem 4.2.64, Pg. I-216 in the book.

4.4 Problem 4.2.67 and Problem 4.2.69, Pg. I-217 in the book.

4.5 Continuity of the derivative operator on test function spaces

- (a) If the sequence of smooth functions ϕ_j converges to 0 in $C^\infty(\mathbb{R})$, show that the sequence of derivatives ϕ'_j also converges to zero in $C^\infty(\mathbb{R})$.
- (b) If the sequence of smooth functions ϕ_j converges to 0 in $\mathcal{S}(\mathbb{R})$, show that the sequence of derivatives ϕ'_j also converges to zero in $\mathcal{S}(\mathbb{R})$.

4.6 The translation operator

Let χ denote any of the usual test function spaces. Define the translation operator τ_y on a smooth function ϕ by $\tau_y(\phi) = \psi \Leftrightarrow \psi(x) = \phi(x - y)$.

- (a) For any given $a \in \mathbb{R}^n$, show that τ_a is a continuous linear map from χ into itself.
- (b) For any given $\phi \in \chi$, show that the map $y \mapsto \tau_y(\phi)$ is continuous, and for any fixed $x \in \mathbb{R}^n$, the map $y \mapsto [\tau_y(\phi)](x)$ is also in χ .
- (c) If $T : \chi \rightarrow \mathbb{R}$ is a distribution, show that the operator $\tau_y^*[T]$ defined by $\tau_y^*[T](\phi) = T(\tau_{-y}(\phi))$ is also a distribution.
- (d) If the distribution T is obtained from a function F with the usual identification $F \mapsto T_F$, then $\tau_y^*[T_F] = T_{\tau_y(F)}$.

4.7 Show that the mapping T_j defined by

$$T_j(\varphi) = \int_{\mathbb{R}} \min(j, -\log(|x|)) \varphi(x) dx$$

is a distribution on the Schwartz space $\mathcal{S}(\mathbb{R})$.

Show that the sequence T_j converges in the sense of distributions on $\mathcal{S}(\mathbb{R})'$ and compute the derivative of the limit distribution.