Math 527B – Spring 08
Homework 8 : Due Apr. 1

8.1 Problems 1.3.14 and 1.3.11, pg. II-307.

8.2 Problems 1.3.12 and 1.3.13, pg. II-307.

8.3 Problems 1.3.16, 1.3.17 and 1.3.18, pg. 307–308.

8.4 In class, we sketched an argument to show that every bounded sequence $\{x_n\}$ in $\mathbb{R}$ has a convergent subsequence by showing that there exists a subsequence $\{x_{n_k}\}$ that converges to $l = \sup(E)$ where

$$E = \{y \in \mathbb{R} \,|\, x_n > y \text{ for infinitely many indices } n \in \mathbb{N}\ \}$$

(a) Show, with all the details, that there is a subsequence $\{x_{n'_k}\}$ that converges to $l' = \inf(E')$ where

$$E' = \{y \in \mathbb{R} \,|\, x_n < y \text{ for infinitely many indices } n \in \mathbb{N}\ \}$$

(b) Using completeness of $\mathbb{R}$ show that, for any bounded sequence $\{x_n\}$ in $\mathbb{R}$, the following limits exist:

$$\limsup\{x_n\} \equiv \lim_{k\to\infty} \sup_{n\geq k} x_n \qquad \liminf\{x_n\} \equiv \lim_{k\to\infty} \sup_{n\geq k} x_n.$$

(c) Show directly from the above definitions that there exist subsequences x_{n_k} and $x_{n'_k}$ such that $x_{n_k} \to \limsup x_n$ and $x_{n'_k} \to \liminf x_n$.

(d) $\{x_n\}$ is a bounded sequence, and $\limsup x_n = L$. Show that, for all $\epsilon > 0$, there exists an $N < \infty$ such that $x_n < L + \epsilon$ for all $n \geq N$.

(e) If $\liminf x_n \geq \limsup x_n$, show that x_n converges. Conversely, if x_n converges, show that $\liminf x_n = \lim x_n = \limsup x_n$.

(f) A point $z \in \mathbb{R}$ is an *accumulation point* or equivalently a *cluster point* or a *limit point* of sequence $\{x_n\}$ is there is a subsequence x_{n_k} such that $x_{n_k} \to z$. if z is an accumulation point for abounded sequence $\{x_n\}$ show that

$$\liminf x_n \leq z \leq \limsup x_n.$$

(g) If l and l' are as defined at the beginning of this problem, show that $l = \limsup x_n$ and $l' = \liminf x_n$.

8.5 Optional A non-negative sequence x_n is subadditive if $x_{n+m} \leq x_n + x_m$ for all $n, m \in \mathbb{N}$. If x_n is a subadditive sequence, show that $\lim_{n\to\infty} \dfrac{x_n}{n}$ exists.

8.6 Accumulation points: $\{x_n\}$ is a given sequence in a topoolgical space A (not necessarily a metric space).

(a) Is this statement true or false – If $y \in \overline{\{x_n\}}$, then y is an accumulation point for the sequence $\{x_n\}$. Prove or give a counterexample.

(b) Let F_n denote the close set $\overline{\{x_n, x_{n+1}, x_{n+2}, \ldots\}}$. Show that, if $y \in F = \cap_{n=1}^{\infty}$, if and only if there is a subsequence x_{n_k} converging to y. (Note: We are not necessarily in a metric space, so you should use the abstract topological definition of convergence.)

(c) If the topological space A is compact, show that every sequence in A has a convergent subsequence. (Hint: Finite intersection property. See also, Problem 1.3.33, Pg. II-313)

8.7 Problem 1.3.34, Pg. II-313.

8.8 Problem 1.3.36, Pg. II-313.