Math 527B – Spring 09
Homework 9 : Due Apr. 15

8.1 (M, d) and (N, ρ) are metric spaces. Show that $f : M \to N$ is uniformly continuous if an only if for every Cauchy sequence $x_n \in M$, $f(x_n)$ is a Cauchy sequence in N.

8.2 $f : [0, 1] \to \mathbb{R}$ is continuous and $f(1) = 1$. $g : [1, \infty) \to \mathbb{R}$ is uniformly continuous and $g(1) = 1$. Define a function $h : [0, \infty) \to \mathbb{R}$ by

$$h(x) = \begin{cases} f(x) & x < 1 \\ 1 & x = 1 \\ g(x) & x > 1 \end{cases}$$

Show that h is uniformly continuous.

8.3 $f : [0, \infty) \to \mathbb{R}$ is bounded and continuous.

(a) Does it follow that f is uniformly continuous. Prove or give a counterexample.

(b) If in addition we are given that $\lim_{x \to \infty} f(x)$ exists, show that f is uniformly continuous.

8.4 Problems 1.4.13 and 1.4.14. Pg. II-317.

8.5 Problems 1.4.41 and 1.4.42. Pg. II-332.

8.6 Problems 1.4.45 and 1.4.48, Pg. II-333.

8.7 Problems 1.4.49 and 1.4.50, pg. II-333.

8.8 $f(x, t)$ is continuous in x and t. Further, the (Riemann) integral $\int_0^\infty f(x, t)dt$ converges uniformly for $x \in [-1, 1]$ to a function $F(x)$. Show that F is continuous on $[-1, 1]$.

8.9 Define

$$f(\omega, t) = \begin{cases} \frac{\sin(\omega t)}{t} & t \neq 0 \\ \omega & t = 0 \end{cases}$$

(a) Show that f is continuous in t and ω.

Let the Riemann integral define

$$I(\omega) = \int_0^\infty f(\omega, t)dt$$

(b) Define what it means for the integral to converge uniformly for $\omega \in [-1, 1]$.

(c) Evaluate $I(\omega)$ for $\omega \in [-1, 1]$. Does the integral converge uniformly?

(d) Show that there exists a continuous function $g : [-1, 1] \times \mathbb{R} \to \mathbb{R}$ such that the Riemann integral defines

$$H(\omega) = \int_0^\infty g(\omega, t)dt$$

for all $\omega \in [-1, 1]$ but $H(\omega)$ has a dense set of discontinuities.