Exam 2

Math 527b – Principles of Analysis II

All parts have equal weight.

(a) Show that there is a unique continuous function on $[0,1]$ which solves the functional equation

$$\mu(x) = \frac{2}{3}\mu(1-x^3) + \frac{\sin(\pi x)}{3} \qquad \forall x \in [0,1]$$

(Hint: Contraction mapping!)

(b) Let $\phi : \mathbb{R} \to \mathbb{R}^2$ be the function $x \mapsto \frac{1}{1+x^2}(2x, 1-x^2)$. An easy calculation shows that

$$d(x,y) \equiv \|\phi(x) - \phi(y)\|_2 = \frac{2|x-y|}{\sqrt{(1+x^2)(1+y^2)}}.$$

Show that $(\mathbb{Q}, d)$ is a metric space, and find (with justification) the completion of $\mathbb{Q}$ with respect to this metric. (Hint: What is $\|\phi(x)\|_2$?)

(c) $f_n : (X, \mathcal{B}) \to \mathbb{R}$ is a sequence of measurable functions. Show that

$$\{x | \lim_k \inf_{n \geq k} f_n(x) < t\} = \bigcup_{j=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \left\{ x \middle| f_n(x) < t - \frac{1}{j} \right\}$$

and use this to conclude that $f(x) = \liminf_n f_n(x)$ is a measurable function.

(d)

$$A = \left\{ \mathbf{x} \in l^2(\mathbb{R}, \mathbb{N}) \middle| \sum_{i=1}^{\infty} 2^i \, |x_i|^2 \leq 1 \right\}.$$

Show that A is compact in the metric topology on l^2.

(e) Compute the following limit or show that it doesn't exist

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left(1 + \frac{k}{n^2}\right)^n$$

Carefully justify all your steps, making sure you state the names of any convergence theorems that you use.