Lévy flights in fluid flows with no KAM surfaces.

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We investigate the Lévy flights observed experimentally in the transport of tracers in a temporally irregular flow that has no KAM surfaces, and show that the Lévy flights are due to the sticking of the tracers near the walls. The tracer is found to spread superdiffusively with an exponent $\nu = 3/2$ in reasonable agreement with the experiments.

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A random walk where the distribution of the individual step sizes has a power law tail and an infinite variance is called a *Lévy flight*. In contrast to ordinary diffusive processes, the growth of the variance $\sigma^2(t)$ of an ensemble of random walkers that undergo Lévy flights can be *anomalous*, so that $\sigma^2(t) \sim t^\nu$, with $\nu \neq 1$. The case $\nu < 1$ is called *subdiffusive* and the case $\nu > 1$ is called *superdiffusive*. A question of considerable interest is the extent to which the Lévy flights have a role in the description of physical phenomena [1,2].

One setting in which Lévy flights occur is the anomalous diffusion seen in transport by two dimensional (2D) area preserving maps and Hamiltonian systems. In these systems, it is observed that chaotic orbits stick near the KAM (Kolmogorov-Arnold-Moser) boundary between the chaotic and the regular regions, and the probability distribution of the sticking times has a power law tail [3]. This power law sticking can lead to superdiffusive behavior [4]. Passive particle advection by an incompressible 2D fluid flow is Hamiltonian and it is therefore expected that 2D flows with coexisting regular (KAM) regions and chaotic regions will lead to anomalous transport of passive tracers [5].

The transport of a passive tracer in various fluid flow regimes has been studied in a recent series of experiments [6]- [9]. In these experiments, fluid is pumped through an annular tank which is rotated. This sets up an essentially 2D flow with vortices in the interior of the tank. As observed in this frame, the flow exhibits jets in the regions near the (moving) walls. The trajectories of neutrally buoyant tracers immersed in the fluid are tracked by a video camera co-rotating with the average speed of the vortices. The variance of the azimuthal coordinate of the tracers and the distribution of the times that a tracer spends in the jets and the vortices are measured experimentally.

Depending on external parameters of the experiment, various flow regimes can be accessed, two of which display superdiffusion of tracers. In the first such regime, the flow is time periodic. In this case, a stroboscopic map of the tracer positions at the temporal period of the flow yields a 2D area preserving map, and an available explanation for the superdiffusion is the mechanism of sticking to KAM surfaces [4]. In the other regime, the flow is temporally irregular but spatially smooth. As discussed subsequently, in a temporally irregular flow, there can be no KAM surfaces. Nevertheless, superdiffusion is still observed with an exponent $\nu = 1.55 \pm 0.25$ [8]. In this regime, the distribution of residence times in the jets decays algebraically with an exponent $\mu \approx 2.6$, demonstrating that the tracers undergo Lévy flights. Since there are no KAM surfaces in this flow, the mechanism leading to Lévy flights remains to be resolved. Our purpose in this letter is to analyze a model of the experimental flow and show that the superdiffusive behavior can be due to the "sticking" of the tracers near the walls of the flow where the flow satisfies a no-slip boundary condition.

We model the experimental flow by an incompressible, temporally irregular and spatially smooth flow in a straight channel ($-\infty < x < \infty$ and $|y| \leq a$) with stationary walls at $y = \pm a$. In terms of the experiment, x is analogous to the azimuthal coordinate and y is analogous to the radial coordinate. The flow is given by a time dependent velocity field $\boldsymbol{v} = (v_x(x,y,t), v_y(x,y,t))$ in the frame in which the walls are stationary. The temporal variations have a correlation time τ_c which is the coherence time of the spatial structures in the flow. The time averaged flow $\langle \boldsymbol{v} \rangle$ at each point is longitudinal (along the x-direction) so that $\langle \boldsymbol{v}(x,y,t)\rangle = (v(y), 0)$ [10]. The time averaged velocity is assumed to have nonzero shear, so that $v(y)$ is not a constant across the channel. The mixing transverse to the average flow (y-direction) is entirely due to the temporally irregular variations in the velocity field $\boldsymbol{v}$. Dispersion in the x direction results from the interplay between the mixing in the y direction and the advection in the x direction by the spatially inhomogenous mean flow. This problem is analogous to that of the longitudinal dispersion of a tracer in a steady shear flow originally studied by Taylor [11], in which case the mixing across the streamlines is due to microscopic diffusion. In this latter case, the transport in the longitudinal

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direction is normal diffusion, $\nu = 1$.

The key feature of our model is the hypothesis that there is an appreciable region near the walls where the longitudinal velocity goes to zero at the wall (no-slip) approximately linearly with distance from the wall, *i.e.*, near $y = \pm a$ we have that v_x is approximately given by the first term in its Taylor series about $|y| = a$, $v_x \approx (a - |y|) f_\pm(x,t)$, where $f_\pm(x,t)$ are spatially smooth functions that vary on the time scale τ_c. Incompressibility implies that $v_y = -\int (\partial v_x/\partial x)\, dy$ so that

$$v_y \approx \pm(1/2)(\partial f_\pm(x,t)/\partial x)(a - |y|)^2, \tag{1}$$

near the walls so that the velocity transverse to the wall goes to zero quadratically in the distance from the walls. At a given position x, v_y is temporally irregular and for time scales longer than τ_c, a particle near a wall executes a random walk in the transverse direction. Near the walls, we can model the transport in the transverse direction by diffusion with a diffusion coefficient $D = \langle v_y^2 \rangle \tau_c$. Eq. (1) implies,

$$D(y) \sim (a - |y|)^4. \tag{2}$$

The diffusion coefficient $D(y)$ approaches zero very rapidly near the walls. Consequently, tracers that get close to the walls will remain near the walls for a long time before they diffuse away. We refer to this behavior as "sticking" to the walls.

We model the distribution of the tracers by a density $p(x,y,t)$ representing the number of tracer particles per unit area in a small area element containing (x,y) at time t. The tracer flux $\mathbf{\Gamma}$ is given by $\mathbf{\Gamma} = p\boldsymbol{v}$, where $\boldsymbol{v}$ is the time dependent flow velocity. The average velocity is longitudinal and its contibution to the flux is $v(y)p(x,y,t)\hat{\boldsymbol{e}}_x$. We model the mixing in the transverse direction as diffusion in y with a y dependent diffusion coefficient $D(y)$ satisfying (2) near the walls. This is a good model for the transport in the flow near the walls, but it does not accurately reflect the transport of tracer in the interior of the flow (because the step size $\sqrt{D\tau_c} = \langle v_y^2 \rangle^{1/2} \tau_c$ can be $O(a)$ in the central channel region). However, this should not matter for our scaling results because the anomalous transport that we obtain is due to the sticking of the tracers near the walls (we return to this point subsequently). With these considerations, we write the flux $\mathbf{\Gamma}$ as

$$\mathbf{\Gamma} = v(y)p(x,y,t)\hat{\boldsymbol{e}}_x - D(y)\frac{\partial p(x,y,t)}{\partial y}\hat{\boldsymbol{e}}_y. \tag{3}$$

The conservation of the total amount of tracer implies that

$$\frac{\partial p}{\partial t} = -\nabla \cdot \mathbf{\Gamma} = \frac{\partial}{\partial y}\left[D(y)\frac{\partial p}{\partial y}\right] - v(y)\frac{\partial p}{\partial x}. \tag{4}$$

The flow is confined by the walls to the region $|y| < a$ and we have no-flux in the y-direction as boundary conditions at $y = \pm a$. We also impose the boundary conditions $p(x,y,t) \to 0$ as $x \to \pm\infty$. An analysis of Eq. (4) with these boundary conditions is carried out in [12]. It is found that $\sigma^2(t) \sim t^\nu$ with the anomalous diffusion exponent $\nu = 3/2$. We also obtain that the distribution of the times the tracers are in the central region of the channel ($|y| < b$) decays exponentially and the distribution of times when the tracers are near the walls ($b < |y| < a$) decays algebraically as $t^{-\mu}$ with $\mu = 5/2$ (independent of the choice of b).

As previously remarked, diffusion is a good model for the transport near the boundaries but not in the interior of the flow. In order to verify that the anomalous diffusion is due to the sticking of the tracers near the walls and that the results obtained from the diffusion model in Eq. (4) are valid, we numerically analyze a model temporally irregular incompressible flow with the appropriate boundary conditions.

This can be done in a computationally efficient way if we can obtain maps $T_n : (x,y) \to (x',y')$ where (x',y') are the coordinates at time $t = n+1$ of a fluid element that was at (x,y) at time $t = n$, so that we do not have to numerically integrate the trajectories of all the tracer particles in a flow. For the numerical simulation, we take $a = 1$ and $\tau_c < 1$.

Note that since the flow $\boldsymbol{v}(x,y,t)$ is temporally irregular (in particular, not periodic), the time dependence of $\boldsymbol{v}$ in the interval $n \le t < (n+1)$ is different from that in the interval $(n+1) \le t < (n+2)$. Therefore, the maps for the positions through the two intervals (*i.e.*, T_n and T_{n+1}) are different and this implies that KAM surfaces (or indeed any invariant sets) are absent, since a set invariant under T_n will not be invariant under T_{n+1}. Since we presume that the time dependence of $\boldsymbol{v}(x,y,t)$ is chaotic, we model the sequence of maps $\{T_n\}$ as varying with n in a stochastic manner [13].

The map T_n is generated by an incompressible flow with no-slip boundary conditions. Therefore, it is required to have the following properties: (1) The map T_n should be area-preserving and (2) It should have no-slip, *i.e.*, if $y = \pm 1$, $T_n(x,y) = (x,y)$.

We exploit the analogy between 2D incompressible flows and Hamiltonian systems [14] to obtain the sequence of maps $\{T_n\}$. A Hamiltonian flow generates a *canonical transformation* and the map T_n is consequently a canonical transformation. Canonical transformations can be obtained using the technique of *generating functions*. If $F_n(x',y)$ is an arbitrary function of the coordinate x' at time $n+1$ and the coordinate y at time n, the map $T_n(x,y) \to (x',y')$ defined implicitly by

$$x = \partial F_n(x',y)/\partial y, \quad y' = \partial F_n(x',y)/\partial x', \tag{5}$$

is a canonical transformation and automatically area-preserving. Here, x' is implicitly defined in terms of x and y, and we need to invert the equation for x in terms of x' and y to obtain the map $(x,y) \to (x',y')$. The map T_n for our model flow is the composition of the maps obtained from the generating functions $F_a(x',y) = x'y - \lambda(y - y^3/3)$ and $F_b(x',y) = x'y + \rho/(2\pi)\cos(2\pi(x'+r_n))(1-y^2)^2$. The map generated by F_a gives the time averaged longitudinal flow $x_{n+1} = x_n + \lambda(1-y_n^2)$ and the map generated by F_b gives a temporally irregular velocity field modulated by the time dependent quantities r_n. T_n is defined implicitly by

$$x_n = x_{n+1} - (1-y_n^2)(\lambda + \frac{2\rho}{\pi} y_n \cos[2\pi(x_{n+1}+r_n)]),$$
$$y_{n+1} = y_n - \rho(1-y_n^2)^2 \sin[2\pi(x_{n+1}+r_n)]. \quad (6)$$

Note that if $y_n = \pm 1$, T_n satisfies the no-slip condition $x_{n+1} = x_n$. Eqns. (6) constitute the numerical model we use to study the transport of tracer in a bounded temporally irregular flow. Our reason for using the form of the generating function in (5) is that it allows us to impose the boundary conditions exactly (albeit at the expense of making the map implicit).

In our simulations, we set $\lambda = 1$, $\rho = 0.3$. We also choose r_n on each time step to be a random sample uniformly distributed in $[0,1)$ (This corresponds to our previously stated choice $\tau_c < 1$). We invert the equation for x_n in terms of x_{n+1} and y_n using Newton's method. We follow 22,000 tracer particles that are initially uniformly distributed in the region $-0.5 < x < 0.5$ and $-1 < y < 1$. The mean position $\overline{x}_n$ is the average over i of $x_n(i)$, where $x_n(i)$ is the x coordinate of the ith tracer particle at the time step n. The variance σ_n^2 of the tracer distribution is then calculated as

$$\sigma_n^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_n(i) - \overline{x}_n)^2. \quad (7)$$

Fig. 1 is a log-log plot of the variance of the tracer distribution at different time steps. The inset shows the slope of the plot of the variance. The slope is initially near $\nu = 2$ but then crosses over to a plateau near $\nu = 1.5$ for large times. (Note the logarithmic time scale in the inset of Fig. 1 implying that ν is near 1.5 most of the time.)

We have also tested the predictions that the distribution of lifetimes in the central region is exponential and in the region near the walls decays algebraically. We choose a threshold velocity $v_{th} = 2/3$ (the average velocity of the transverse flow) and a time interval $T_0 = 10$. We assign a tracer to state 1 (the central region) in the time interval $nT_0 \le t < (n+1)T_0$ if the average velocity over T_0 steps $(x((n+1)T_0) - x(nT_0))/T_0 > v_{th}$, and to state 2 otherwise. (The basic result is independent of v_{th}.)

Fig. 2 shows a numerically obtained distribution of the lifetimes in state 1. The solid line is a fit. The distribution of lifetimes is exponential with a time constant which is given by the slope of the straight line in the linear-log plot.

Fig. 3 shows a numerically obtained distribution of the lifetimes for the events that correspond to the sticking near the walls in the numerical model. The solid line has the theoretically predicted long-time, power law slope of $-5/2 = -\mu$. The data are roughly consistent with the theoretical prediction in the long time limit.

In Refs. [7,8], analysis of the data from the temporally irregular flow regime yields a value of the anomalous diffusion exponent of $\nu = 1.55 \pm 0.25$ [8] in good agreement with our theoretical result of $\nu = 3/2$. Their experimental exponent for state 2 (motion near the walls) is $\mu \approx 2.6$ [8] in agreement with our theoretical result $\mu = 5/2$ [15]. For state 1, our theoretical model has exponential decay of the lifetime distribution. In Ref. [8] the authors comment that the distribution is "perhaps power law at intermediate times with crossover to exponential behavior at longer times but the data is inconclusive."

In conclusion, we have studied the longitudinal dispersion of a passive scalar in a bounded temporally irregular shear flow with no KAM surfaces in this letter. We find that Lévy flights and superdiffusive behavior can arise as a consequence of tracers sticking to the walls due to the boundary conditions of the flow. We obtain an anomalous spreading exponent $\nu = 3/2$ in reasonable agreement with recent experiments.

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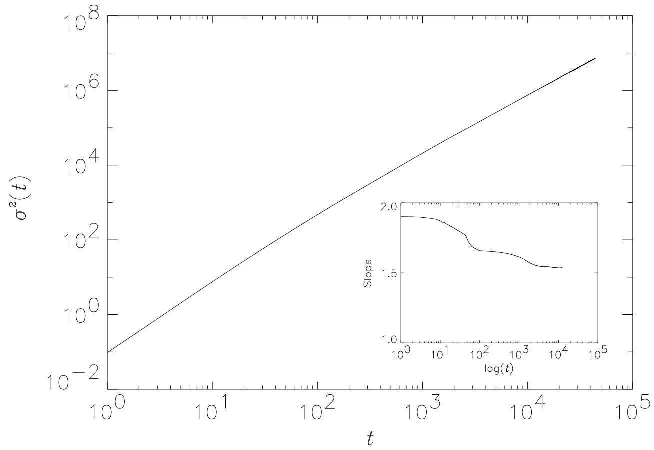

FIG. 1. The variance of the tracer distribution for the numerical model flow. The inset shows the slope of the plot of the variance.

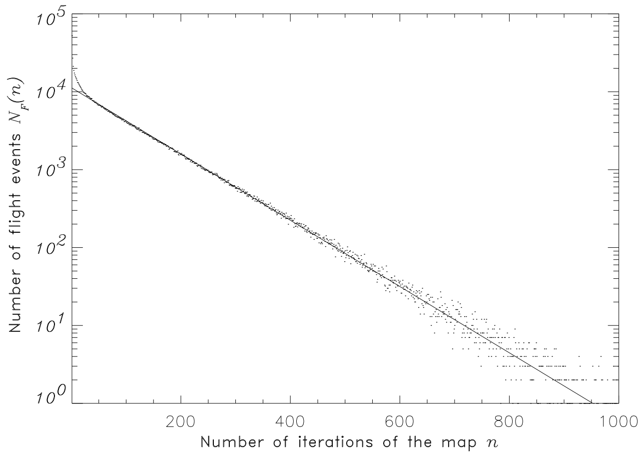

FIG. 2. The distribution of lifetimes when the tracers are not stuck to the walls. The solid line is a fit.

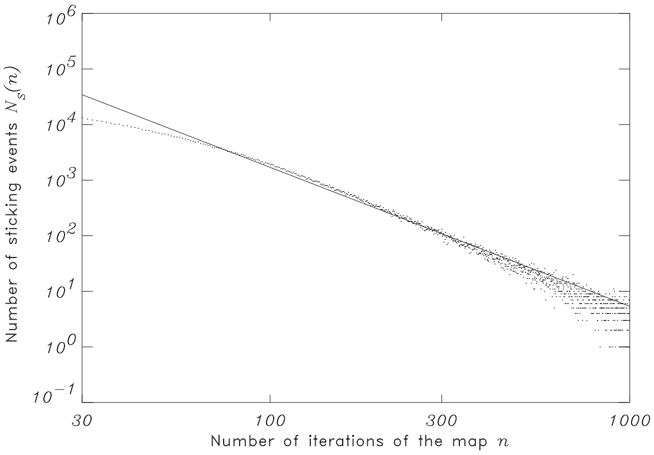

FIG. 3. The distribution of sticking times. The solid line has the theoretical slope of -5/2.