MULTIPLE SCALE BEHAVIORS IN NONCONVEX VARIATIONAL PROBLEMS

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1. Introduction

In a variety of condensed matter systems, multiple scale structures, both *singularities* and *microstructure* arise spontaneously as a result of the system attempting to find a ground state (energy minimum) [41]. This behavior is ubiquitous and it occurs in a wide variety of problems including type-II superconductors, liquid crystals, nonlinear elasticity, micromagnetics, and thin elastic sheets.

In all these situations, the small scale structures are much smaller than the macroscopic scale, but are much larger than an atomic scale. We therefore expect that this behavior can be understood purely in terms of continuum models. It is of considerable interest to identify the mechanisms leading to singularities/microstructure in condensed matter systems. An important feature of these systems that display multiple scale behavior is the presence of multiple, often widely disparate, energy scales, associated with different physical mechanisms.

For example, the "thinness" of an elastic sheet is reflected in the ratio of the flexural and the in-plane rigidities of the sheet, *i.e*, for thin sheets, the bending energy scale is much smaller than the stretching energy scale.

A common argument for the occurrence of singularities/microstructure in systems with multiple energy scales is the following:

1. The "weak" energy scale involves a higher order derivative than the strong energy scale.
2. The "weak" energy is a singular perturbation in the Euler–Lagrange equations for the system.
3. The singular perturbation results in singular/small scale structures in the solutions to the equations, and these structures can be analyzed using matched asymptotics.

The last item in this list also suggests an approach to study the small scale structures that arise from energy minimization. In the context of thin sheets, this approach has been used to study the the vertex singularities [10], and the elastic ridges [35].

The presence of a singular perturbation in the Euler-Lagrange equations is by itself not a guarantee for the occurrence of multiple scale behavior in the solutions. An illustrative example of this is the following comparison between a thin elastic rod

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that is confined by a ring in two dimensions, and a thin elastic sheet that is confined by a sphere[1]. The rod "curls up" parallel to itself touching the boundary. The maximum curvature that occurs in the solution is on the same scale as the forcing, *viz.* the curvature of the confining ring. The two dimensional sheet, on the other hand, *crumples*, and has vertices whose length scale is the order of the thickness t of the sheet, and ridges whose width is of the order $t^{1/3}L^{2/3}$, where L is the radius of the confining sphere. For both the rod, and for the sheet, the bending energy gives rise to a term with a 4th order derivative and a small coefficient, which is a singular perturbation of the Euler–Lagrange equation without the bending energy. This example shows that the presence of multiple energy scales/singular perturbation, does not in itself create multiple scale behavior. The geometry of the problem plays a significant role in determining the structures that arise spontaneously.

Our main goal in this paper is to investigate this relation between geometry and the spontaneous formation of small scale structures. Rather than work with the Euler–Lagrange equation, we will work directly with the underlying *variational problem.* The relevant energy $\mathcal{E}[u]$ of a configuration u is typically an integral of a *local* energy density $W(x, u, Du, \dots)$ over the domain $\mathcal{S}$:

$$\text{(VP)} \qquad \mathcal{E}[u] = \int_{\mathcal{S}} W(x, u, Du, \dots)dx.$$

Here x is a coordinate on the domain, the configuration of the system is given by a function $u(x)$, and $Du, D^2u, \dots$ denote the derivatives of u. We denote the admissible set of configurations for the given boundary conditions by $\mathcal{A}$. The goal is to find the minimum energy configuration $u^* \in \mathcal{A}$ where the minimum of $\mathcal{E}[u]$ over $u \in \mathcal{A}$ is attained.

A natural approach is to consider a minimizing sequence $u_k \in \mathcal{A}$, *i.e.*, a sequence of admissible configurations u_k such that $\mathcal{E}[u_k]$ decreases to the minimum value of the energy (more properly the infimum $\inf_{u\in\mathcal{A}} \mathcal{E}[u]$). The hope is that the sequence u_k (or possibly a subsequence) converges to an admissible configuration $u^* \in \mathcal{A}$, and this u^* gives the ground state for the energy $\mathcal{E}$.

The key structural condition that guarantees the existence of a minimizer in $\mathcal{A}$ is *convexity.* If W is convex in the highest order derivative among it's arguments, and the set $\mathcal{A}$ is convex, the variational problem is said to be *convex.* In this case, u_k (or a subsequence) converges to a minimizer u^* [22, 14]. In fact, there are different notions of convexity for variational problems. These notions have subtle differences and are applicable in different contexts [41].

If W is nonconvex (in the appropriate sense), the minimizing sequence u_k might exhibit oscillatory behavior, or concentrate the energy into small scale structures, on *increasingly finer scales*, as $k \to \infty$ [41]. In this case, no subsequence of u_k converges in an appropriate (strong) sense, and the variational problem (VP) need not have a minimizer. A example of this phenomenon is the classical Bolza problem [58, 1]

$$\text{Minimize } I_a = \int_0^1 \left[(u_x^2 - 1)^2 + au^2\right] dx, \qquad \text{subject to } u(0) = u(1) = 0.$$

[1]This example was related to us by Tom Witten

Note that the energy density is nonconvex in u_x. If $a = 0$, any piecewise linear function with slopes ± 1 is a minimizer for I. Thus, there is a large degeneracy in the set of minimizers. If $a = 1$, minimizing sequences u_k consist of sawtooth functions with slope ± 1 and amplitudes decaying to zero. The functions u_k develop increasingly finer oscillations as $k \to \infty$. In particular, the derivatives u_{kx} do not converge pointwise (and hence strongly), and the sequence u_k converge (weakly) to $u(x) \equiv 0$, which is not a minimizer for the problem [58].

Nonconvexity thus provides a natural mechanism for the formation of small scale structures from energy minimization. Nonconvex variational problems, however, have the following "unphysical" features, that real systems cannot exhibit –

1. Structures can form on arbitrarily fine scales.
2. The minimization problem may not have a solution at all!
3. There are a large number of macroscopically distinct low energy states, that arise from different minimizing sequences.

These problems are "cured" if we consider perturbed energies $\mathcal{E}^\epsilon$ that include a small scale cutoff arising from a "convex" energy. Let us consider the functional form,

$$(\mathrm{VP}^\epsilon) \qquad \mathcal{E}^\epsilon[u] = \int_{\mathcal{S}} W(x, u, Du)dx + \epsilon^2 \int_{\mathcal{S}} F(x, u, Du, D^2u)dx.$$

where W is nonconvex in Du, and F is *convex* in D^2u. For $\epsilon = 0$, we recover the "bare" nonconvex problem, that might generate structures on arbitrarily fine scales. For any nonzero ϵ however, the second term penalizes structures on very fine scales. This effectively introduces a small scale cutoff through a singular perturbation and regularizes the original nonconvex variational problem (VP). For every $\epsilon \neq 0$, $\mathcal{E}^\epsilon$ has a minimizer u^ϵ. In addition, the singular perturbation is a selection mechanism, and it can select a small subset of all the low energy states of $\mathcal{E}$ as the "true" minimizers. For an illustration of this phenomenon, see for example Ref. [51].

The elastic energy of thin 2-dimensional sheets embedded in $\mathbb{R}^3$ falls in this framework. Let $(\mathcal{S}, g)$ be a two-dimensional Riemannian manifold with a metric g [49, 56]. The admissible set $\mathcal{A}$ consists of immersions $u : \mathcal{S} \to \Omega \subset \mathbb{R}^n$, possibly subject to additional boundary conditions. The energy of a configuration u is given by the sum of the stretching and the bending energies via

$$(\mathrm{GC}) \qquad \mathcal{E}^\epsilon[u] = \int_{\mathcal{S}} \|Du^{\mathrm{T}} \cdot Du - g\|^2 dx + \epsilon^2 \int_{\mathcal{S}} \|P_{\mathbf{n}}[D^2u] - \hat{K}\|^2 dx,$$

where $\hat{K}$ is the extrinsic curvature of the reference (zero energy) configuration of the sheet. If $x^i, i = 1, 2$ are coordinates in the surface, and $u^\alpha, \alpha = 1, 2, 3$ are coordinates in the ambient space, Du denotes the matrix of partial derivatives $\partial u^\alpha / \partial x^i$, and D^2u denotes the second partial derivatives. $P_{\mathbf{n}}$ is the orthogonal projection along the normal to the immersed surface $u[\mathcal{S}] \in \mathbb{R}^3$, so that $P_{\mathbf{n}}[D^2u]$ is the extrinsic curvature of the configuration u [49]. Finally, ϵ is the (scaled) thickness of the sheet.

For $\epsilon = 0$, we get a stretching energy functional that penalizes deviations from isometry, since u is isometric if and only if $Du^T \cdot Du = g$. This functional is *nonconvex*, since a function with $Du = 0$ belongs to the convex hull of the set $K = \{u \mid Du^T \cdot Du = g\}$, but u with $Du = 0$ itself does not belong to K.

$\mathcal{E}^\epsilon$ regularizes this nonconvex stretching energy by penalizing large curvatures through the bending energy. The ϵ^2 (= square of the thickness) multiplying the bending energy reflects the disparity in the scale of the stretching and the bending energies. This energy functional describes many physical systems including crumpled elastic sheets [57, 35, 5, 10], delamination of thin-films deposited on substrates [45, 29] and multiple scale bucking in free sheets [47, 48]. A modified functional incorporating a non-zero Poisson's ratio also describes the "wrinkling" of sheets under tension [12, 11].

2. Multiple scale behaviors in thin elastic sheets

The variational problem (GC) generates singularities for appropriate boundary conditions. This has been investigated experimentally [13], by asymptotic analysis [10, 35] and by numerical simulations [35, 32]. Two distinct types of singular structures – point-like vertex singularities [10] are at the ends of line-like ridge singularities have been identified [5, 35, 36]. These singularities are analogous to the points and the creases in crumpled paper [5]. Both the vertices [10, 9, 40] and the ridges[35, 18, 19] have been thoroughly investigated in Physics literature. Asymptotic analysis suggests that, in the limit $\epsilon \to 0$, most of the energy is concentrated on the ridges [35].

Figure. 1 is a schematic representation of a ridge. In the limit $\epsilon \to 0$, the ridge becomes a line across which there is a constant jump in the gradient Du. However, for any nonzero ϵ, the ridge has additional structure along it's length. Asymptotic analysis [35] and scaling arguments [57, 34] suggest that the width of the region over which the gradient changes, as well as the region over which the energy concentrates, are not uniform along the length of the ridge. Rather, the width of this region is given by $d(x) \lesssim C\epsilon^{1/3}x^{2/3}$, where x is the distance to the closest endpoint of the ridge [35]. This is illustrated in Fig. 1.

In Refs. [55, 53], we prove that

Theorem 1. *The energy $\mathcal{E}^\epsilon$ of a ridge in a crumpled sheet has the scaling law*

$$c\epsilon^{5/3}L^{1/3} \leq \mathcal{E}^\epsilon[u^\epsilon] \leq C\left(\epsilon^{5/3}L^{1/3} + \epsilon^2 \log\left(\frac{\epsilon^{1/3}L^{2/3}}{a}\right)\right)$$

where L is the length of the ridge and a is the scale on which the vertex at the end of the ridge is smooth. Further, the energy concentrates on a region whose width is given by $\rho(x) \leq C\max(a, \epsilon^{1/3}x^{2/5}L^{4/15})$, where x is the distance from the (closest) endpoint of the ridge.

The nontrivial structure along the length of the ridge singularity is reflected in the scaling for the energy ($L^{1/3}$ instead of L) and the scaling for $\rho(x)$ with an explicit dependence on x.

Minimizing sequences for (GC) can also develop very fine scale oscillations [30]. Ben Belgacem *et al.* have shown that [6, 7], for a compressed thin film

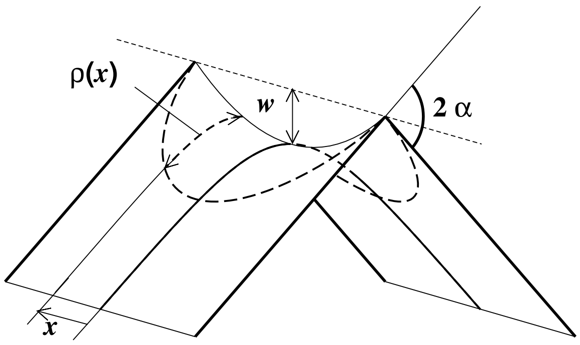

FIGURE 1. A minimal ridge. The boundary conditions are given by a frame (the thick solid lines) bent through an angle. The sheet is essentially flat outside the region bounded by the two dashed curves, and the bulk of the energy is concentrated in this region.

Theorem 2. *The energy of the minimizer satisfies*

$$c\lambda^{3/2}\epsilon L \leq \mathcal{E}^{\epsilon}[u^{\epsilon}] \leq C\lambda^{3/2}\epsilon L,$$

where L is a typical length scale of the domain, and λ is the compression factor.

A construction for the upper bound strongly suggests that the minimizers develop an infinitely branched network with oscillations on increasingly finer scales as $\epsilon \to 0$. The (out of plane) oscillations have a wavelength and an amplitude that scale as $\epsilon^{\mu}x^{(1-\mu)}$ where $\frac{1}{4} < \mu < \frac{1}{3}$ and x is the distance from the boundary [7]. The oscillation scale therefore goes to zero for *all* x as $\epsilon \to 0$. In the limit, the minimizers converge pointwise to a limiting configuration u^* which is *not strain free*. This is analogous to the situation for the Bolza problem, where the pointwise limit of a minimizing sequence is not a minimizer for the problem.

Combining these results with the observation that a one-dimensional rod confined in two dimensions does not show any singular behavior as $\epsilon \to 0$, we are led to define the following three types of limiting behaviors as $\epsilon \to 0$.

Non singular limit: As $\epsilon \to 0$, the minimizers do not develop any singular/small scale structures. The stretching energy remains small for all ϵ, and $\int F dx$ remains bounded for all ϵ.

Concentration: As $\epsilon \to 0$, the minimizers converge to a limiting configuration that is *singular* (not smooth). Although the minimizer is not smooth, there is a well defined derivative almost everywhere, and the stretching energy of the limiting configuration is zero. Also, $\int F dx$ diverges as $\epsilon \to 0$.

Oscillation: As $\epsilon \to 0$, the minimizers converge to a limiting configuration u^* that is *not strain free*. The derivatives Du^{ϵ} do not converge pointwise to Du^*. Rather they oscillate about Du^* as $\epsilon \to 0$. $\int F dx$ again diverges in this limit.

Our physical intuition is that the behavior gets increasingly singular as we go down in the above list, and this leads to an increase in the asymptotic energy as $\epsilon \to 0$. A consequence will be the following criteria for deciding the $\epsilon \to 0$ behavior of the thin sheet.

1. If the imposed boundary conditions allow a strain free configuration $\tilde{u}$, with $\int F dx$ finite, the $\epsilon \to 0$ behavior of the minimizers will be non singular.
2. If the boundary conditions allow a strain free configuration $\tilde{u}$, but not one with a finite $\int F dx$, the minimizers will concentrate energy on (a set which is "similar" to) the set on which $D^2\tilde{u}$ is singular.
3. If there is *no strain free* configuration compatible with the boundary conditions, the minimizers will display *microstructure* (small scale oscillations) as $\epsilon \to 0$.

Let us emphasize that these arguments purely suggestive, and not mathematically rigorous, although the first condition can be made rigorous using recent results by Pakzad [46]. These arguments address the form of the ground states of $\mathcal{E}^\epsilon$ for small ϵ, but not the scaling behavior of the energy or the length scales associated with the minimizer (See Q1 and Q2 below). However, knowing the expected structure of the minimizers greatly aids in the rigorous analysis of these scaling laws.

Finally we note that, where applicable, these criteria have reduced the "hard" analytic problem of determining the form of the minimizers of $\mathcal{E}^\epsilon$ to a simpler geometric problem of the existence of strain free configurations that satisfy the given boundary conditions.

3. The geometry of variational problems

We now return to considering the variational problem for

$$\mathcal{E}^\epsilon = \int W(x,u,Du)dx + \epsilon^2 \int F(x,u,Du,D^2u)dx$$

where W is nonconvex in Du and F is convex in D^2u. W and F are normalized such that they are non-negative, and the minimum value that they attain is zero. Let K_x denote the set of all (u, Du) such that $W(x,u,Du) = 0$ a.e. We assume that K_x is smooth and bounded for all x and doesn't change (or changes smoothly) with x. We will say $u \in K$ if $(u, Du) \in K_x$ for a.e. x. Also W and F satisfy the growth conditions

$$W(x,u,Du) \geq C[\text{dist}((u,Du),K_x)]^\alpha, \qquad F(x,u,Du,D^2u) \geq \|D^2u\|^m,\ m>1.$$

Finally, we assume that the energy of the minimizer goes to zero with ϵ, *i.e.*, $\lim_{\epsilon\to 0} \mathcal{E}^\epsilon(u^\epsilon) = 0$.

As we saw above, the following questions arise in studying the variational problem for small ϵ –

Q1 – Morphology of the solutions: Do the minimizers u^ϵ display oscillations or concentration effects? Describe the oscillatory/singular regions in the solutions.

Q2 – Scaling laws: How do $\mathcal{E}^\epsilon(u^\epsilon)$, and the length scales associated with the oscillations/singular regions in the minimizer u^ϵ depend on ϵ?

Q3 – Variational Convergence: Does $\lim_{\epsilon \to 0} u^\epsilon$ exist, in some appropriate sense? If so, is it possible to define a suitable limiting process for the energy functionals, so that $\lim_{\epsilon\to 0} \mathcal{E}^\epsilon \to \bar{\mathcal{E}}$, implies that $\bar{\mathcal{E}}$ has a minimizer $\lim_{\epsilon \to 0} u^\epsilon$?

The first two questions are of direct physical interest. The third question is closely related to the notion of Γ – convergence [16, 15]. In situations where we can define a limiting energy, it allows us us to identify the asymptotic energy of "singular" objects, *e.g*, the ridges and the vertices in a crumpled sheet. These objects can now be thought of as the "elementary excitations" for the limiting problem in the following sense. The limiting energy yields a reduced description of the full variational problem, where the entire energy can be written as a sum of the energies of the various singular structures.

If ϵ is small and $\mathcal{E}^\epsilon(u^\epsilon) \approx 0$, it follows that $W(x, u^\epsilon, \ldots)$, is small over a large part of the domain. This motivates the consideration of the set

$$\mathcal{A}' \equiv \mathcal{A}'(\emptyset) = \left\{ u \in \mathcal{A} \cap K, \int_{\mathcal{S}} F dx < \infty \right\}.$$

If $\mathcal{A}'$ is nonempty there exist functions in K which have bounded $\int F$, and are admissible. Our arguments at the end of the last section would therefore suggest that the limit is *nonsingular*. We can show that this is indeed the case.

In particular, if $\mathcal{A}'$ is nonempty and convex, we can show that, as $\epsilon \to 0$, the solutions u^ϵ converge to the minimizer of the convex variational problem – Minimize $\int F dx$ for $u \in \mathcal{A}'$ [54]. Note that we have removed all reference to the small parameter ϵ in the minimization. In this case, the minimizer is "uniformly smooth" in the sense that

$$\int F(x, u^\epsilon, Du^\epsilon, D^2 u^\epsilon) \leq J < \infty \quad \text{uniformly as } \epsilon \to 0.$$

We therefore do not expect the development of small scale structures for small ϵ. Also, the asymptotic energy of the minimizers is $\mathcal{E}^\epsilon(u^\epsilon) \sim \epsilon^2$.

If $\mathcal{A}' \neq \emptyset$, we have completely answered questions Q1–Q3 from above. The limit is nonsingular, the energy scales as ϵ^2, and the limit problem is

$$\text{Minimize} \int F(x, u, Du, D^2 u) dx \quad \text{over} \quad u \in \mathcal{A} \cap K.$$

If $\mathcal{A}' = \emptyset$, the minimizers u^ϵ *necessarily* develop either singularities or microstructure as $\epsilon \to 0$. We now look at this situation in more detail.

If the minimizing sequence u^ϵ does not contain fine scale oscillations, then u^ϵ and Du^ϵ still converge pointwise a.e. to the set $W = 0$. However, there is now an exceptional set of points, the *singular set* $\mathcal{R}$, where some of the higher derivatives diverge in a manner such that

$$\limsup_{\epsilon \to 0} \int_O F dx = +\infty,$$

for any open set $O \supset \mathcal{R}$.

This is a signature of energy concentrating on the set $\mathcal{R}$ as $\epsilon \to 0$. Motivated by the example of a crumpled sheet, we expect that Du^ϵ, which converges pointwise a.e by assumption, converges to a function Du^* that has bounded variation (henceforth written as $Du^* \in BV$). This is essentially the statement that we can still make sense of the second derivative D^2u^* in the sense of measures [59]. The set $\mathcal{R}$ is precisely the *singular support* of the measure D^2u^*.

Since energy does not concentrate at points in $\mathcal{S} - \mathcal{R}$, the previous argument restricted to $\mathcal{S} - \mathcal{R}$ gives

$$\mathcal{A}'(\mathcal{R}) = \left\{ u \in \mathcal{A} \cap K, Du \in BV, \int_{\mathcal{S}-O} F dx < \infty \quad \forall O \supset \mathcal{R}, \right\} \neq \emptyset.$$

We will call a set $\mathcal{R}$ with $\mathcal{A}'(\mathcal{R}) \neq \emptyset$ an *obstruction.* The idea behind this terminology is that by removing (A small neighborhood of) the set $\mathcal{R}$ from the domain $\mathcal{S}$, we can find an admissible configuration u that is in K, and has $\int F dx$ finite. This idea was used in Ref. [20] to identify the relevant singular structures, purely in terms of the local energy density.

We define $\mathcal{R}$ to be a *minimal obstruction*, if $\mathcal{A}'(\mathcal{R}) \neq \emptyset$, and no proper subset of $\mathcal{R}$ has this property. Our arguments from above suggest the natural conjecture

Conjecture 1. *If the energy does concentrate in the limit $\epsilon \to 0$, we expect that the energy will concentrate on a (finite union of) minimal obstruction(s).*

If $u^* \in \mathcal{A} \cap K$, and $Du^* \in BV$, it follows that $\mathcal{R}^*$, the singular support of D^2u^* is a set with finite H^1 measure (length/perimeter in 2 dimensions) [59]. Using the growth condition for F, we can show that

$$\mathcal{E}^\epsilon(u^\epsilon) \leq C\epsilon^{2/m}\mathrm{Per}(\mathcal{R}^*)$$

where Per denotes the Perimeter [54]. We do not expect that this result is sharp. In particular, it is not sharp for the minimal ridge in an elastic sheet, where the energy scales with an exponent

$$\frac{5}{3} > \frac{2}{m} = 1.$$

Finally, we consider situations which lead to oscillations. In this case, Du^ϵ does not converge pointwise a.e. Rather it only converges weakly and it generates an associated Young Measure in the limit $\epsilon \to 0$. In this case, D^2u^ϵ does not converge in any reasonable sense, not even as a measure. This leads to

Conjecture 2. *If $\mathcal{E}^\epsilon(u^\epsilon) \to 0$, as $\epsilon \to 0$, a necessary and sufficient condition for the occurrence of oscillations in the limit is $\mathcal{B}' = \{u \in \mathcal{A} \cap K, Du \in BV\} = \emptyset$.*

We investigate the above conjectures in Ref. [54]. The above conjectures and arguments addresses item Q1 in the list above. Through these arguments, we have related analytical questions about the family of variational problems (VP$^\epsilon$) to *geometric/topological* questions about the set of functions $u \in K$ with the additional conditions $\int F dx < \infty$ or $Du \in BV$. We should think of these conditions as *regularity* (smoothness) requirements on solutions to the *Differential Inclusion* $u \in K$, *i.e.*, $(u, Du) \in K_x$ a.e.

4. Isometric immersions and Convex Integration

For the elastic energy (GC), $W = 0$ implies u is an isometric immersion. In Ref. [52], we investigated the existence of smooth isometric immersions of the unit disk in $\mathbb{R}^m$ into a ball B_r^d in $\mathbb{R}^d$ with radius r. These boundary conditions are relevant to the *confinement* problem, where an elastic sheet (here m-dimensional) is crushed inside a d dimensional ball (making r small).

We showed following "rigidity" result [52] about the existence of immersions $\phi : \mathcal{S} \to \Omega$

Theorem 3. *If $d \geq 2m$, we always have a smooth isometric immersion $\phi : \mathcal{S} \to \Omega$.*
If $d < 2m$, for every S there exists an $r > 0$ such that there exists no smooth (or even C^3) isometric immersion $\phi : \mathcal{S} \to B_r^d$, a d dimensional ball with radius r.

This shows that the minimizers will be non singular for $d \geq 2m$ and will be singular otherwise. In contrast, classical results of Nash and Kuiper [43, 33] show that a sufficient condition for the existence of C^1 isometries is $d \geq m + 1$.

We wish to apply our arguments from above to deduce the morphology of the minimizers as $\epsilon \to 0$. For our arguments to apply, we first need to show that $\mathcal{E}^\epsilon(u^\epsilon) \to 0$. We prove this result in Ref. [54]. The key idea is that the Nash-Kuiper theorem gives us C^1 isometric immersions that satisfy the boundary conditions. By an appropriate smoothing of these immersions, *away from the boundaries*, on a scale that depends on ϵ, we can obtain near isometries $\tilde{u}^\epsilon$ whose stretching energy is small, and $\int F$ diverges slower that ϵ^{-2}. Consequently, $\mathcal{E}^\epsilon(u^\epsilon) \leq \int [W[\tilde{u}^\epsilon] + \epsilon^2 F[\tilde{u}^\epsilon]]dx \to 0$ as $\epsilon \to 0$.

We can now apply our arguments to deduce the morphology of the minimizers. Our proof in [52] identifies the mechanism behind the "rigidity" of the isometric immersions, and allows us to identify sets that can be excised to remove the rigidity [20]. Refining Thm. 3 to identify these obstructions, we conjecture a lower bound of the dimension of a minimal obstructions for periodic boundary conditions [20] –

Conjecture 3. *The minimal obstruction has dimension at least $2m - d - 1$. If $d = m + 1, m \geq 2$, the minimal obstruction has dimension $m - 1$.*

Armed with this knowledge, we can find solutions u^* such that $u^* \in \mathcal{A} \cap K$, and $Du \in BV$. Numerical simulations of (GC) showed energy concentration on singular sets consistent with predictions this conjecture. Numerics also suggest that the lower bounds in this conjecture are sharp [20].

In our identification of the minimal obstructions [20], we only consider solutions that are *smooth* (or at least C^3) isometric immersions. However, our definition of $\mathcal{A}'$ required us to consider all solutions with $\int F dx < \infty$. This is a gap in the analysis since we do not *a priori* know that every isometric immersion with $\int F dx < \infty$ is "close" to a C^3 immersion. Pakzad [46] has proved this result for 2-sheets embedded in 3-dimensions. This allows us to fill in the gap in the analysis, for two dimensional sheets confined in $\mathbb{R}^3$. However, conjecture 3 is still open in other situations.

For the compression boundary conditions that are relevant to the delamination of thin-films, it is easy to see that $\mathcal{B} = \{u \in \mathcal{A} \cap K, Du \in BV\} = \emptyset$. Indeed, the trace theorem for BV functions [59] implies that, for $(u, Du) \in K_x$ and $Du \in BV$, we have $(u, Du) \in K_x$ at the boundary. This is clearly not allowed by the boundary conditions. As we expect, the minimizers do show oscillatory behavior as $\epsilon \to 0$ [6, 7].

Note that, independent of Theorem 2, the Nash-Kuiper theorem already implies $\mathcal{E}^\epsilon(u^\epsilon) \to 0$ by the same arguments as above [54]. In general, if we can find C^1 solutions to the differential inclusion

$$(u, Du) \in K_x, \qquad u \in \mathcal{A},$$

the smoothing argument from above will show that $E^\epsilon(u^\epsilon) \to 0$. Thus it is of considerable interest to solve these differential inclusions, which are really *under-determined PDEs.*

A very useful tool for this purpose is Gromov's theory of *convex integration* [24, 50]. Convex integration addresses questions in geometric rigidity and under-determined PDE, and it gives a method for constructing C^1 solutions to differential inclusions $(u, Du) \in K_x$ [50]. These solutions have oscillations on an infinite number of scales. They are obtained by approximately solving the problem on a coarse scale, and then recursively refining the solution by adding oscillations on increasingly finer scales [50]. This is precisely the idea underlying the proofs of Nash [43] and Kuiper [33].

In the context of Isometric immersions, adding oscillations can only "increase" the metric tensor g. Therefore, the approximate solution at each stage should be *short* [24]. An immersion $u : \mathcal{S} \to \Omega$ is short, if the distance between any two points in the space Ω, is less than or equal to the distance between these points in the reference manifold $(\mathcal{S}, g)$. Since Ω is a subset of an Euclidean space with a Euclidean metric, an immersion is short if

$$\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\alpha}{\partial x^j} v^i v^j \leq g(v, v)$$

for all vectors v tangent to $\mathcal{S}$. The embedding is *strictly short*, if the above inequality is strict, *i.e.*, the strain induced by the embedding is compressive everywhere. We can think of the convex integration procedure as starting with a compressed manifold, and then relaxing the compressive strain by adding oscillations on increasingly finer scales.

Gromov shows that, starting with any strictly short embedding of a m-dimensional manifold in a $m+1$ dimensional space, we can add appropriate oscillations to obtain a C^1 isometric immersion, *that is close to the initial short embedding* [24]. Our smoothing argument now implies that any set of boundary conditions that admits a *short* immersion, has $\mathcal{E}^\epsilon(u^\epsilon) \to 0$ [54]. In particular, the confinement boundary conditions, periodic boundary conditions, and compressive boundary conditions all admit short embeddings, and our criteria for determining the morphology of the minimizers are therefore applicable. Free elastic sheets, *i.e.*, elastic sheets with no imposed boundary conditions [47] also admit short immersion. We will consider such sheets in the next section.

If K is connected, the convex integration procedure gives exact C^1 solutions to $(u, Du) \in K_x$, *i.e.*, solutions to the (nonregularized) problem (VP). Müller and Šverák [42] have adapted this technique to generate Lipschitz solutions u in situations where K is not connected, *i.e*, multi-well problems.

It would be very useful to connect these convex integration solutions to minimizers of the regularized problem for $\mathcal{E}^\epsilon$. Convex integration solutions have oscillations on all scales, this might provide a natural mechanism for generating the multiple scale oscillations that are observed in a variety of situations, including delamination of thin films [23], buckling of free sheets [47, 48], Austenite-Martensite phase boundaries [31, 8] and Magnetic microstructures [17].

Since the convex integration solutions solve the noregularized problem (VP), they reflect some of the weaknesses of the formulation (VP), which motivated our consideration of the perturbed problem (VP$^\epsilon$).

1. There is a large degeneracy of the exact solutions.
2. Convex integration typically yields solution that are very "rough" ($\int F = \infty$, or $Du \notin BV$). This however doesn't imply that there are no smoother solutions (say $Du \in BV$).

These features reflect the multi-scale nature of the convex integration solutions. To apply this construction to the perturbed problem (VP$^\epsilon$) we need to incorporate the selection mechanism, and the small scale cutoff, due to the perturbation into the convex integration procedure. How to do this remains a largely open question at the present time.

5. Free sheets

The multiple scale buckling on the edges of leaves and in ripped sheets have been investigated both experimentally [47, 48] and through a combination of analytic [44, 37, 38, 3, 48, 4] and numerical methods [37, 3, 4].

In Ref. [44], the authors argue that an appropriate model for the multiple scale buckling is the minimization of an elastic energy for a hyperbolic plane $\mathbb{H}^2$ (a surface of constant negative Gaussian curvature) embedded in $\mathbb{R}^3$. In Ref. [44], the authors discretize the hyperbolic plane through a "covering" Cayley tree [25], and then construct isometric immersions of the the Cayley tree into $\mathbb{R}^3$. The isometric embeddings of these graphs show oscillations on a scale that gets finer as we move away from the center of the surface, and the authors conclude that this is a possible mechanism for the multiple scale oscillations in the edges of leaves [44].

In Ref. [38, 3], the authors consider the buckling of a thin annular strip with imposed boundary conditions. This is connected with the problem of elasticity of negatively curved surfaces, since such surfaces can be constructed by appropriately gluing together such annular strips [26]. In Ref. [4], the authors study the elasticity of a negatively curved surface by linearizing about a fiducial (smooth) isometric embedding of the center-plane, and show that the competition between the bending and the stretching energies can lead to a *self-similar* buckling cascade.

The problem of the isometric immersion of the Hyperbolic plane $\mathbb{H}^2$ into $\mathbb{R}^3$ has a long history. For a delightful review of this subject, see [26]. Hilbert proved that there is no real analytic isometric immersion of $\mathbb{H}^2$ onto a complete subset of $\mathbb{R}^3$ [27]. This implies there is at least one singular point in an embedding of $\mathbb{H}^2$ in $\mathbb{R}^3$. This theorem was improved by Holmgren [28] who showed that, given a (local) smooth embedding of $\mathbb{H}^2$ in $\mathbb{R}^3$, the embedding cannot be extended isometrically and smoothly beyond is a finite distance d. Unfortunately, d depends on the local embedding, and there isn't a uniform bound for the size of the "largest" piece of the hyperbolic plane that can be isometrically embedded in $\mathbb{R}^3$.

Hilbert's theorem was improved by Amsler [2] who showed that every sufficiently smooth immersion of the hyperbolic plane into $\mathbb{R}^3$ has a singular "edge", *i.e*, a one-dimensional submanifold beyond which the embedding is no longer smooth. In 1964, Efimov [21, 39] proved that there is no C^2 isometric embedding of $\mathbb{H}^2$, or any complete manifold with curvature that is uniformly negative in $\mathbb{R}^3$. However, we know from Kuiper's theorem [33], that there is indeed a C^1 embedding of $\mathbb{H}^2$ in any subset of $\mathbb{R}^3$.

To our knowledge, it is still an open question whether or not there is an embedding of $\mathbb{H}^2$ in $\mathbb{R}^3$ such that the gradient of the embedding is a BV function. On the other hand the results due to Pakzad [46], suggest that there are no embeddings with $\int F < \infty$, or with $Du \in BV$, if there is no smooth embedding.

There are two distinct mechanisms for the spontaneous development of structure in thin "hyperbolic" sheets. For "small" hyperbolic sheets, there are smooth (C^2) isometric immersions in $\mathbb{R}^3$. These immersions might themselves have oscillations [44, 4]. However, they satisfy $\int F < \infty$. For such sheets, the limit $\epsilon \to 0$ is *non-singular* in our classification, *i.e*, the length scales associated with the oscillations do not change as $\epsilon \to 0$, and the minimum energy configurations can be obtained by minimizing the bending energy over all the isometric embeddings of the sheet [37, 3]. In this case, the limiting immersion should be smooth (have a well defined curvature everywhere).

In the experiments in Ref. [47], the sheets have an additional (implicit) boundary condition that the sheet is flat far away from the tear. In this circumstance, we expect that a version of Amsler's theorem applies, and for a sheet with a sufficiently large zone with plastic deformation, there is a *singular edge* beyond which we cannot extend the isometric immersion smoothly [54]. In fact, as we argue above, we do not expect that we can extend the sheet isometrically keeping $Du \in BV$. This will again lead to oscillation phenomena in the limit $\epsilon \to 0$. However, these oscillations are of a different character, and this mechanism will have the following experimental signatures

1. The energy will decay slower than ϵ^2, and the structures cannot be explained by a purely bending theory. In particular, the stretching and the bending energies stay comparable as $\epsilon \to 0$.
2. As $\epsilon \to 0$, the minimizers converge to a *short* embedding weakly, and the limiting configuration is *not isometric* (strain free). This requires that the amplitude of the oscillations decay no faster than the wavelength as the edge is approached [12, 54]. Also, there is no meaningful second derivative for the limiting configuration. This behavior is different from the behavior in Ref. [4], where the authors remark that the limiting configuration of the sheet is an isometric embedding.
3. The length scales associated with the oscillations scale with ϵ, and are not constant. Rather they should go to zero as $\epsilon \to 0$. A theory based on only considering isometric immersions of the Hyperbolic plane will not see this effect.

Some of these features are indeed observed experimentally, and they are reported in Ref. [48]. In particular, the wavelength and the amplitude of the oscillations scale as $\epsilon^{1/3}$ in the experiment [48].

6. Local and Nonlocal singularities

We have seen many examples where a minimizing sequence fails to converge strongly, and results in either energy concentrating on small sets, or the system developing small scale oscillations. Depending on whether or not the small scales depend on the distance to a boundary, we can crudely classify the different ways in which a minimizing sequence fails to converge strongly as in Table 1.

Behaviors	Concentration	Oscillations
Local	Vertices in crumpled sheets Superconducting vortices	Buckling in free sheets Binary Alloys
Nonlocal	Ridges in crumpled sheets	Delamination of Thin films Austenite-Martensite phase boundaries

Table 1. Examples displaying different scenarios by which minimizing sequences fail to converge strongly, and lead to multiple scale behaviors.

Interestingly Theorems 1 and 2 imply that the crumpling problem has solutions that are, nonlocal, are of fundamentally different characters, and depend on boundary conditions. This runs contrary to the "folklore" that singularities are robust and universal "local" phenomena independent of the precise form of boundary conditions.

It is easy to construct boundary conditions[2] for a thin elastic sheet such that the allowed configurations include both the "blistering" type solutions from Refs. [6, 7], and "crumpled" configurations where the energy concentrates on ridges and vertices. The energy of the blistering solution depends on a parameter λ which measures the effective compression due to the boundary conditions and scales as $\lambda^{3/2}\epsilon L$. The energy of the corresponding crumpled sheet on the other hand scales as $\epsilon^{5/3}L^{1/3}$, and is essentially independent of λ if λ is large. Consequently, with a one parameter family of boundary conditions, we can move from a situation where the oscillatory "blistering" solutions are energetically preferred to a situation where the "crumpling" solutions are energetically preferable. This suggests the intriguing possibility of *Global transitions*, where the nature of the solution (oscillations vs. concentration) changes, and these transitions are mediated by the boundary conditions. This possibility needs to be investigated further.

In conclusion, we have proposed a set of criteria which enable us to study/predict the complex morphology of ground states for condensed matter systems, through the analysis of an associated, simpler geometric problem. We have used these criteria to investigate the multiple scale behaviors seen in free elastic sheets [47, 48], and obtain very good agreement with experimental observations. We also propose a classification

[2]This possibility was first suggested to us by Stefan Müller

for the types of multiple scale behaviors that arise out of energy minimization, and in particular we note the possibility of small scale structures in the bulk of a system being sensitively dependent on the precise nature of the boundary conditions "far away". This behavior is indeed seen in thin elastic sheets. Finally, we also note the possibility of "global transitions" between solutions of different types in our classification, and we expect that such transitions can be observed in thin elastic sheets. We hope this motivates further experimental, theoretical and numerical investigation of these systems, which are in many ways a paradigm for studying multiple scale behaviors.

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References

1. Giovanni Alberti and Stefan Müller, *A new approach to variational problems with multiple scales*, Comm. Pure Appl. Math. **54** (2001), no. 7, 761–825.
2. Marc-Henri Amsler, *Des surfaces à courbure négative constante dans l'espace à trois dimensions et de leurs singularités*, Math. Ann. **130** (1955), 234–256.
3. B. Audoly and A. Boudaoud, *'ruban à godets': an elastic model for ripples in plant leaves*, C. R. Mec. **330** (2002), no. 12, 831–836.
4. B. Audoly and A Boudaoud, *Self-similar structures near boundaries in strained systems*, preprint, 2003.
5. M. Ben Amar and Y. Pomeau, *Crumpled paper*, Proc. Roy. Soc. London Ser. A **453** (1997), 729.
6. H. Ben Belgacem, S. Conti, A. DeSimone, and S. Müller, *Rigorous bounds for the Föppl-von Kármán theory of isotropically compressed plates*, J. Nonlinear Sci. **10** (2000), no. 6, 661–683.
7. ______, *Energy scaling of compressed elastic films - three-dimensional elasticity and reduced theories*, Arch. Rat. Mech. Anal. **164** (2002), 1–37.
8. K. Bhattacharya and R. D. James, *A theory of thin films of martensitic materials with applications to microactuators*, J. Mech. Phys. Solids **47** (1999), no. 3, 531–576.
9. A Boudaoud, P Patricio, Y Couder, and M Ben Amar, *Dynamics of singularities in a constrained elastic plate*, Nature **407** (2000), no. 6805, 718–720.
10. E. Cerda and L. Mahadevan, *Concical surfaces and crescent singularities in crumpled sheets*, Phys. Rev. Lett. **80** (1998), 2354.
11. E. Cerda and L. Mahadevan, *Geometry and physics of wrinkling*, Phys. Rev. Lett. **90** (2003), 074302.
12. E. Cerda, K. Ravi-Chandar, and L. Mahadevan, *Wrinkling of an elastic sheet under tension*, Nature **419** (2002), 579.
13. S. Chaïeb, F. Melo, and J-C. Géminard, *Experimental study of developable cones*, Phys. Rev. Lett. **80** (1998), 2358.
14. Bernard Dacorogna, *Direct methods in the calculus of variations*, Springer-Verlag, Berlin, 1989.
15. Gianni Dal Maso, *An introduction to* Γ*-convergence*, Birkhäuser Boston Inc., Boston, MA, 1993.

16. Ennio De Giorgi and Gianni Dal Maso, *Γ-convergence and calculus of variations*, Mathematical theories of optimization (Genova, 1981), Springer, Berlin, 1983, pp. 121–143.
17. Antonio DeSimone, Robert V. Kohn, Stefan Müller, and Felix Otto, *Magnetic microstructures—a paradigm of multiscale problems*, ICIAM 99 (Edinburgh), Oxford Univ. Press, Oxford, 2000, pp. 175–190.
18. B. A. DiDonna, *Scaling of the buckling transition of ridges in thin sheets*, Phys. Rev. E **66** (2002), no. 1, 016601.
19. B. A. DiDonna and Witten T. A., *Anomalous strength of membranes with elastic ridges*, Phys. Rev. Lett. **87** (2001), no. 20, 206105.
20. B. A. DiDonna, T. A. Witten, S. C. Venkataramani, and E. M. Kramer, *Singularities, structures, and scaling in deformed m-dimensional elastic manifolds*, Phys. Rev. E (3) **65** (2002), no. 1, part 2, 016603, 25.
21. N. V. Efimov, *Generation of singularites on surfaces of negative curvature*, Mat. Sb. (N.S.) **64 (106)** (1964), 286–320.
22. Lawrence C. Evans, *Partial differential equations*, American Mathematical Society, Providence, RI, 1998.
23. G. Gioia, A. DeSimone, M. Ortiz, and A. M. Cuitiño, *Folding energetics in thin-film diaphragms*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **458** (2002), no. 2021, 1223–1229.
24. Mikhael Gromov, *Partial differential relations*, Springer-Verlag, Berlin, 1986.
25. Frank Harary, *Graph theory*, Addison-Wesley Publishing Co., Reading, Mass.-Menlo Park, Calif.-London, 1969.
26. David W. Henderson and Daina Taimina, *Crocheting the hyperbolic plane*, Math. Intelligencer **23** (2001), no. 2, 17–28.
27. David Hilbert, *Ueber Flächen von constanter Gaussscher Krümmung*, Trans. Amer. Math. Soc. **2** (1901), no. 1, 87–99.
28. E. Holmgren, *Sur les surfaces à courbure constante négative*, C.R. Acad. Sci. Paris **134** (1902), 740–743.
29. W. Jin and R. V. Kohn, *Singular perturbation and the energy of folds*, J. Nonlinear Sci. **10** (2000), no. 3, 355–390.
30. Weimin Jin and Peter Sternberg, *Energy estimates for the von Kármán model of thin-film blistering*, J. Math. Phys. **42** (2001), no. 1, 192–199.
31. Robert V. Kohn and Stefan Müller, *Surface energy and microstructure in coherent phase transitions*, Comm. Pure Appl. Math. **47** (1994), no. 4, 405–435.
32. E. M. Kramer and T. A. Witten, *Stress condensation in crushed elastic manifolds*, Phys. Rev. Lett. **78** (1997), 1303.
33. Nicolaas H. Kuiper, *On C^1-isometric imbeddings. I, II*, Nederl. Akad. Wetensch. Proc. Ser. A. **58** = Indag. Math. **17** (1955), 545–556, 683–689.
34. A. Lobkovsky, S. Gentges, H. Li, D. Morse, and T. A. Witten, *Scaling properties of stretching ridges in a crumpled elastic sheet*, Science **270** (1995), 1482.
35. A. E. Lobkovsky, *Boundary layer analysis of the ridge singularity in a thin plate*, Phys. Rev. E. **53** (1996), 3750.
36. A. E. Lobkovsky and T. A. Witten, *Properties of ridges in elastic membranes*, Phys. Rev. E **55** (1997), 1577.
37. M. Marder, *The shape of the edge of a leaf*, preprint `http://arXiv:cond-mat/0208232`, 2002.
38. M. Marder, E. Sharon, S. Smith, and B. Roman, *Theory of Edge of Leaves*, To appear in Europhys. Lett., 2003.
39. Tilla Klotz Milnor, *Efimov's theorem about complete immersed surfaces of negative curvature*, Advances in Math. **8** (1972), 474–543.
40. T. Mora and A. Boudaoud, *Thin elastic plates: On the core of developable cones*, Europhys. Lett. **59** (2002), no. 1, 41–47.
41. Stefan Müller, *Variational models for microstructure and phase transitions*, Calculus of variations and geometric evolution problems (Cetraro, 1996), Springer, Berlin, 1999, pp. 85–210.

42. Stefan Müller and Vladimir Šverák, *Attainment results for the two-well problem by convex integration*, Geometric analysis and the calculus of variations, Internat. Press, Cambridge, MA, 1996, pp. 239–251.
43. John Nash, *C^1 isometric imbeddings*, Ann. of Math. (2) **60** (1954), 383–396.
44. Sergei Nechaev and Raphaël Voituriez, *On the plant leaf's boundary, "jupe à godets" and conformal embeddings*, J. Phys. A **34** (2001), no. 49, 11069–11082.
45. Michael Ortiz and Gustavo Gioia, *The morphology and folding patterns of buckling-driven thin-film blisters*, J. Mech. Phys. Solids **42** (1994), no. 3, 531–559.
46. Mohammad R. Pakzad, *On the sobolev space of isometric immersions*, preprint, 2002.
47. E. Sharon, B. Roman, M. Marder, G-S. Shin, and H. L. Swinney, *Buckling cascade in free thin sheets*, Nature **419** (2002), 579.
48. E. Sharon, S. Smith, H. L. Swinney, and B. Roman, *Geometrically Induced Buckling Cascade in Free Sheets*, preprint, 2003.
49. Michael Spivak, *A comprehensive introduction to differential geometry. Vols. I – V*, second ed., Publish or Perish Inc., Wilmington, Del., 1979.
50. David Spring, *Convex integration theory*, Birkhäuser Verlag, Basel, 1998, Solutions to the h-principle in geometry and topology.
51. Peter Sternberg, *The effect of a singular perturbation on nonconvex variational problems*, Arch. Rational Mech. Anal. **101** (1988), no. 3, 209–260.
52. S. C. Venkataramani, T. A. Witten, E. M. Kramer, and R. P. Geroch, *Limitations on the smooth confinement of an unstretchable manifold*, J. Math. Phys. **41** (2000), no. 7, 5107–5128.
53. Shankar C. Venkataramani, *The energy of crumpled sheets in Föpl-von Karman plate theory.*, preprint.
54. ———, *Geometry and multiple scale behaviors in thin elastic sheets*, in preparation.
55. ———, *Lower bounds for the energy in a crumpled elastic sheet – a minimal ridge*, Submitted to Nonlinearity, Preprint `http://arxiv.org/abs/math.AP/0210012`.
56. Frank W. Warner, *Foundations of differentiable manifolds and Lie groups*, Scott, Foresman and Co., Glenview, Ill.-London, 1971.
57. T. A. Witten and H. Li, *Asymptotic shape of a fullerene ball*, Europhys. Lett. **23** (1993), 51–55.
58. L. C. Young, *Lectures on the calculus of variations and optimal control theory*, W. B. Saunders Co., Philadelphia, 1969.
59. William P. Ziemer, *Weakly differentiable functions*, Graduate Texts in Mathematics, vol. 120, Springer-Verlag, New York, 1989, Sobolev spaces and functions of bounded variation.

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