Blowup in the Chemotaxis equation

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Abstract.

Certain bacteria can drift up the gradient of a substance that they secrete and during this process they also diffuse through space. We study a simple system of partial differential equations for the density of bacteria and the concentration of the substance, that describes both effects. The competition between diffusion and chemotactic attraction leads to rich dynamics for the system and the possible behaviors of the solutions include convergence to time independent solutions or the formation of finite-time singularities where the density diverges. Our goal is to describe the different kinds of solutions that lead to these outcomes and to see the extent to which the outcomes are universal. We restrict our attention to rotationally invariant systems and find that the behavior of the system depends strongly on the space dimension d. For $d < 2$ there is no blow up of the density. For $d > 2$ we find a correspondence between the multiplicity and stability of radially symmetric, time independent solutions of this system on bounded domains and parabolic self-similar blowup solutions on infinite domains. For a range of d, the system can exhibit countably many steady states co-existing with countably many self-similar blowup solutions and solutions leading to non self-similar blowup.

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1. Introduction

Let ρ be a density, perhaps of bacteria, and c the concentration of a substance (attractant) secreted by the bacteria. Both ρ and c are necessarily positive. If the attractant diffuses and is produced by the bacteria, we have

$$\partial_t c - D_c \Delta c = k_1 \rho.$$

Imagine that the bacteria diffuse and are also chemotactically attracted by the attractant. In that case, they tend to move toward regions of higher c, so that the drift velocity proportional to ∇c. The density ρ then obeys

$$\partial_t \rho = D_b \Delta \rho - k_2 \nabla \cdot (\rho \nabla c)$$

If we non-dimensionalize the equations by defining $\tilde{\mathbf{x}} = \mathbf{x}/L, \tilde{t} = D_b t/L^2, \tilde{c} = k_1 c/D_b, \tilde{\rho} = L^2 k_1 k_2 \rho/(D_c D_b)$, and $\Lambda = D_c/D_b$, where L is an arbitrary length scale, the equations reduce to

$$\begin{aligned} \frac{\partial \tilde{\rho}}{\partial \tilde{t}} &= \tilde{\Delta}\tilde{\rho} - \tilde{\nabla} \cdot (\tilde{\rho}\tilde{\nabla}\tilde{c}) \\ \frac{1}{\Lambda}\frac{\partial \tilde{c}}{\partial \tilde{t}} &= \tilde{\Delta}\tilde{c} + \tilde{\rho}. \end{aligned}$$

If we look at the situation where the substance c diffuses much more rapidly than the bacteria, we are interested in the situation $\Lambda \gg 1$. Taking the limit $\Lambda \to \infty$ and dropping the tildes, we get the set of equations

$$\begin{aligned} \partial_t \rho &= \Delta \rho - \nabla \cdot (\rho \nabla c) \\ -\Delta c &= \rho. \end{aligned} \tag{1}$$

The scalar functions $\rho = \rho(\mathbf{x}, t)$ and $c = c(\mathbf{x}, t)$ depend on $\mathbf{x} \in \Omega \subseteq \mathbf{R}^d$ and on $t \geq 0$. We are interested in solving the system in (1) for domains Ω that are bounded as well as the case when $\Omega = \mathbf{R}^d$. For the solutions on a bounded domain, the boundary conditions are zero flux of the scalar ρ through $\partial\Omega$ and the scalar c is zero on $\partial\Omega$. For solutions on $\mathbf{R}^d$, the boundary conditions are decay at infinity in ρ and in c. These equations constitute the Keller-Segel [1] (KS) model and they were introduced in the context of chemotactic aggregation.

Note that we had the freedom to choose the length scale L in the above non-dimensionalization. This implies that the equations in (1) have a scale invariance. We will return to this point subsequently.

We are interested in the question of what is the final state of the system if we start from an initial density distribution. Figures 1 and 2 show simulations which give two different possible final fates: In one, the diffusion wins out and the bacteria spread out so that the system heads for a steady state. In the opposite case, the concentration of the "attractant", c, draws the bacteria together and they achieve an infinite density at some finite time T. We wish to study the possible blowup.

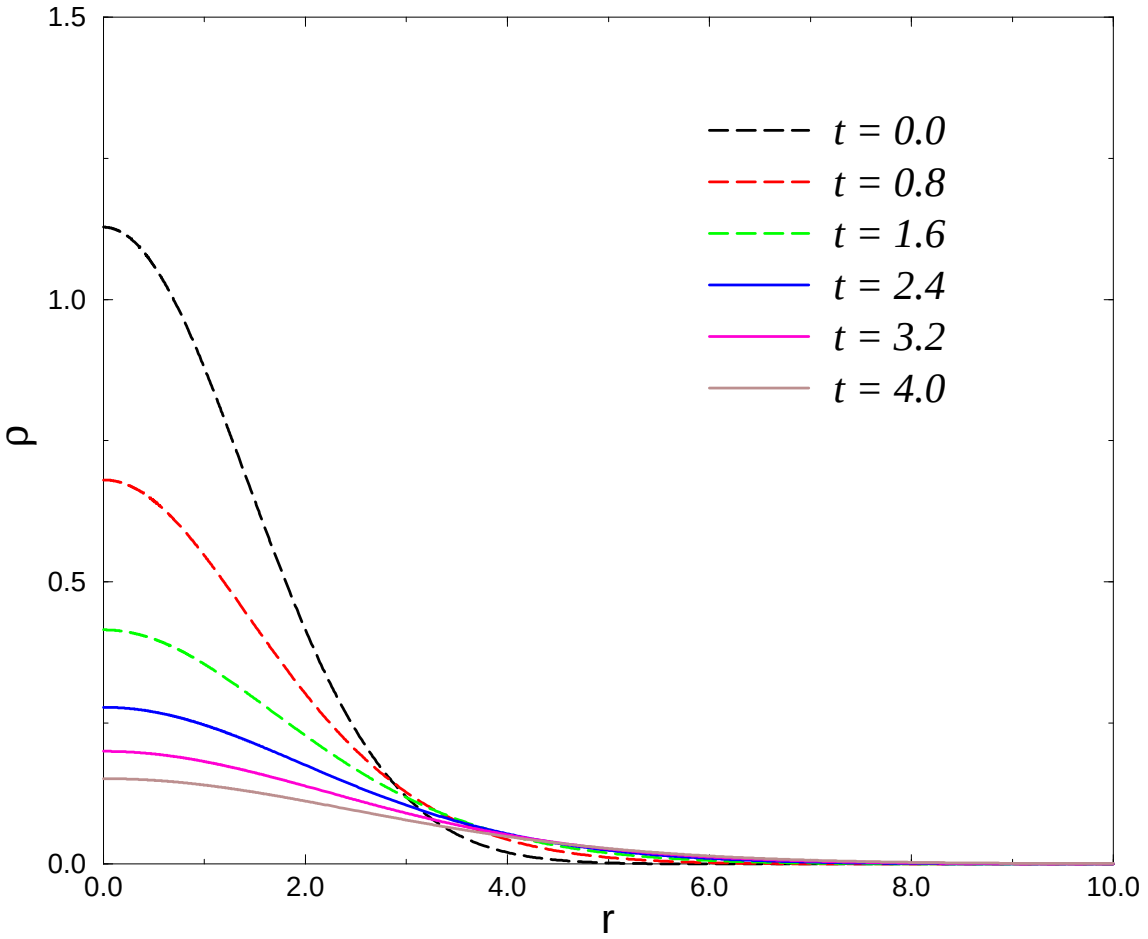

Figure 1. Simulation with spherically symmetric initial data in three dimensions. The solution goes to a time independent density profile for large times

This problem has several experimental realizations. For example, the equations represent the dissipative dynamics of a mass density ρ interacting with itself via a gravitational potential ϕ. In this example blowup corresponds to gravitational collapse at low Reynolds numbers. The time dynamics of gravitational collapse into black holes is a topic of current interest [2]. Another realization occurs in bacterial chemotaxis - Here ρ represents the density of bacteria and c is the concentration of a (rapidly diffusing) attractant that the bacteria secrete. The bacteria drift up the gradient of the attractant. This last example has recently been realized with *E. Coli* [3, 4]. Blowup corresponds to the formation of dense aggregates of bacteria. The experiments show a competition between cylindrically symmetric collapse (in which the final state is a line) and spherically symmetric clumping to a geometrical point.

From the mathematical side, this problem is one of a class of blowup problems [5] in which small scales are produced in a finite time. A canonical example for finite time blowup in parabolic PDEs is the finite time blowup in the solutions of the semi-linear

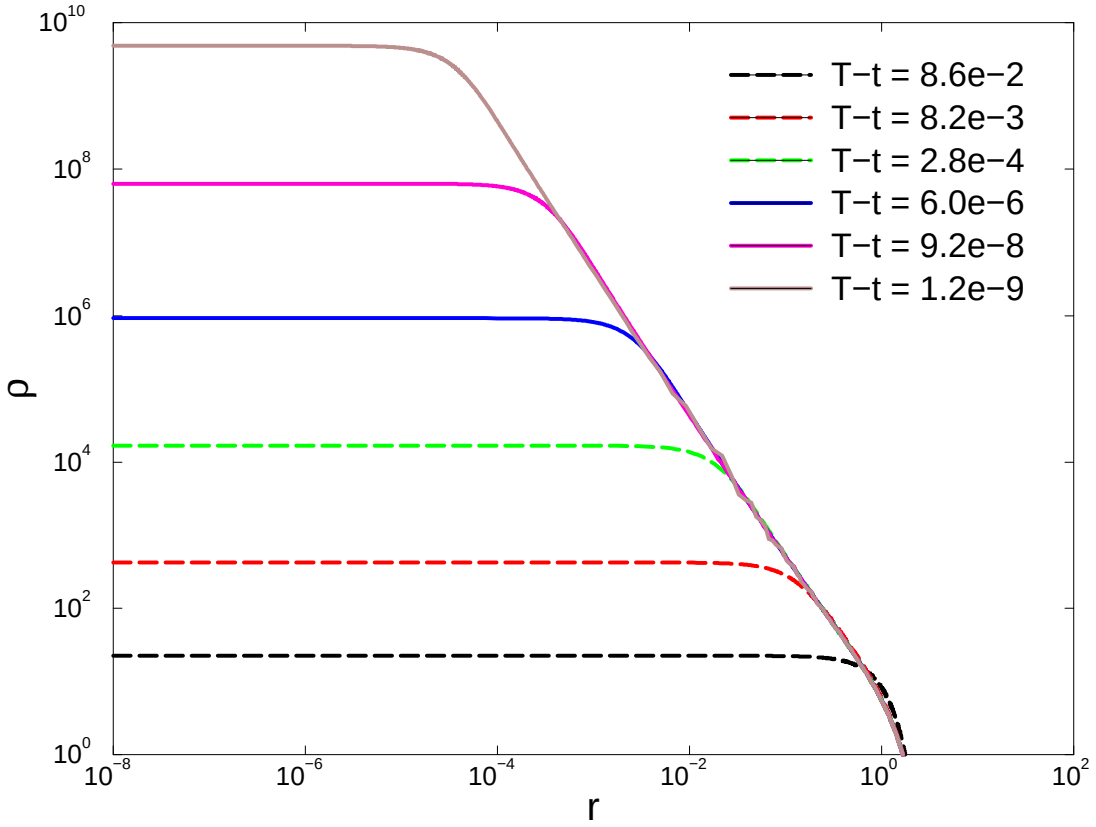

Figure 2. Simulation with spherically symmetric initial data in 3 dimensions. The figure shows a double log plot of blowup at the origin at a finite time T. The maximum density increases rapidly with time signaling the onset of a singularity.

heat equation

$$\partial_t \rho = \Delta \rho + \rho^2.$$

This problem has been studied by various authors, and Bricmont and Kupiainen [6] classify all the possible blowup profiles for this equation. The equation for the density ρ in (1) can be rewritten as

$$\partial_t \rho = \Delta \rho - \nabla c \cdot \nabla \rho + \rho^2,$$

so that these equations are similar to the semi-linear heat equations except for the presence of the advection term $\nabla c \cdot \nabla \rho$. It turns out that the presence of this term significantly alters the properties of the solutions. Unlike the semi-linear heat equation, the mechanism for the blowup in the KS system depends on the space dimension d. Also, as we will see later, the blowup in the KS system proceeds on scales that are different from those in the semilinear heat equation.

Another equation with finite time blowup that has been the subject of much study the Nonlinear Schrödinger equation [7]. The blowup in the Nonlinear Schrödinger equation bears some similarities to the blowup in the KS system, especially in the scaling of the self-similar blowing up solutions and in the dependence of the blowup on the dimensionality of the space.

Blowup of solutions of the KS equations was predicted by Nanjudiah [8] and analyzed in a definitive paper of Childress and Percus [9]. This problem has already been much studied in two dimensions [10]. In most of our work, our approach will be to look at a spherically symmetric d-dimensional geometry. In this situation d is just a parameter in the equations. In this parameter dependence, $d = 2$ is a critical dimension for the problem, since there can be blowup for $d \geq 2$ but all the solutions are regular for $d < 2$.

1.1. Outline of paper

The goal of this paper is to understand the different kinds of solutions of the KS system and the various mechanisms that exist for blowup as a function of space dimension.

In Sec. 2 we deduce the existence of a non-increasing energy functional on solutions in both unbounded and bounded domains. The existence of the energy rules out the possibility of the solutions having recurrent or almost recurrent behavior so that the system has no periodic or chaotic solutions. We also specialize the equations to the rotationally symmetrical case and derive a comparison principle for the solutions which states that if one solution is above another at time zero (in a certain sense) this ordering will persist in subsequent times. The scale-invariance of the equations and the comparison principle serve as major tools of the subsequent work.

In Sec. 3, we characterize the time-independent solutions of our model both in infinite and finite spatial domains. For $2 < d < 10$, we obtain a rich family of steady solutions on bounded domains with the number of steady solutions showing a non-trivial dependence on the total mass of the solution and the radial size of the domain. The stability of the solutions is described. The time-independent solutions are important because they can describe a $t \to \infty$ behavior of the solutions. Since some of the steady solutions are unstable, their stable manifolds serve as separatrices between different types of dynamical behavior.

Section 4 concerns itself with the possible formation of finite time singularities. We show that for $d > 2$ there exist initial conditions that lead to blowup. We also show that for initial data with a finite number of maxima, if the radial locations of the maxima do not oscillate very rapidly as we approach the time of blowup (for instance if the behavior is monotonic very close to the blowup time or if the maxima move with a bounded velocity), the blowup can only occur at the origin. Finally, we demonstrate that all solutions with a bounded initial density distribution are regular in $d < 2$.

In Sec. 5, we discuss various types of solutions that lead to finite time singularities. In Sec. 5.1, we analyze the self-similar solutions of the KS system with the parabolic

scaling (the scaled radial coordinate is $\eta = r/\sqrt{T-t}$ where T is the blowup time) obtained by requiring that the diffusion and the nonlinear terms of the equation be in balance. We show that the system has a countable family of similarity solutions with this scaling for $2 < d < 10$. However, there is only one stable solution which serves as an attractor for solutions with generic initial data. In Sec 5.2, we present results for a different mechanism of blowup in $d > 2$ for generic initial conditions that leads to formation of scales distinct from the parabolic scaling for the self-similar solutions. This is a generalization of solutions considered in [11] for $d = 3$. In Sec 5.3, we exhibit solutions where the diffusion becomes irrelevant in comparison to the nonlinearity as we approach the blowup time.

Section 6 describes the stability and convergence properties of the different self-similar solutions and relates these properties to the results of our numerical simulations. We also briefly discuss the roles of some of the unstable solutions (critical solutions) in determining the nature of the phase space for this system and in determining the relation between the initial data and the final states in this problem.

2. Energy and the comparison principle

In the solutions to Eq. (1) on $\mathbf{R}^d$, the function c has the representation

$$c(\mathbf{x},t) = \int E(\mathbf{x}-\mathbf{y})\rho(\mathbf{y},t)d\mathbf{y}$$

where E is the fundamental solution of $-\Delta$,

$$E(\mathbf{z}) = -c_2 \log(|\mathbf{z}|)$$

in two dimensions and

$$E(\mathbf{z}) = c_d|\mathbf{z}|^{2-d}$$

in $d \geq 3$ dimensions. The first natural question is that of local existence: do smooth initial positive ρ evolve in time for some time, remaining smooth and positive? Positivity follows from the convergence of an iteration

$$(\partial_t - \Delta + \nabla c^{(n-1)} \cdot \nabla)\rho^{(n)} = (\rho^{(n-1)})^2$$

with

$$c^{(n-1)} = E * \rho^{(n-1)}$$

because the linear equation one computes at each level has a minimum principle

$$\min(\rho^{(n)}(\mathbf{x},t)) \geq \min(\rho^{(n)}(\mathbf{x},0)).$$

The equation for ρ in (1) can be written as

$$\partial_t \rho = -\nabla \cdot \Gamma,$$

where the mass-flux Γ is given by

$$\Gamma = -\nabla\rho + \rho\nabla c = -\rho\nabla(\log\rho - c) \tag{2}$$

If the boundary conditions are decay in ρ at infinity for unbounded domains or zero flux on the boundaries for bounded domains, it follows that

$$\int_\Omega \rho(\mathbf{x},t)d\mathbf{x} = \int_\Omega \rho(\mathbf{x},0)d\mathbf{x}.$$

We will denote this conserved quantity by

$$M = M[\rho] = \int_\Omega \rho(\mathbf{x})d\mathbf{x}.$$

Another property of the equations is scale invariance - On unbounded domains, if $\rho(\mathbf{x},t)$ is a solution then, for any $L > 0$,

$$\rho_L(\mathbf{x},t) = L^{-2}\rho(\frac{\mathbf{x}}{L},\frac{t}{L^2})$$

is again a solution. Note that

$$M[\rho_L] = L^{d-2}M[\rho].$$

This foretells a special relationship between M and $d = 2$. More generally

$$\int(\rho_L)^p d\mathbf{x} = L^{d-2p}\int\rho^p d\mathbf{x}$$

which speaks about the importance of $\mathbf{L}^{\frac{d}{2}}(d\mathbf{x})$ in the problem. The equation for the evolution of $\int\rho^p d\mathbf{x}$ is particularly simple:

$$\frac{d}{dt}\int\rho^p(\mathbf{x},t)d\mathbf{x} = -p(p-1)\int|\nabla\rho(\mathbf{x},t)|^2\rho(\mathbf{x},t)^{p-2}d\mathbf{x} + (p-1)\int\rho(\mathbf{x},t)^{p+1} \tag{3}$$

At $p = \frac{d}{2}$ the growth term in the RHS can be bound in terms of the dissipation term by

$$\int\rho^{\frac{d+2}{2}}(\mathbf{x},t)d\mathbf{x} \le C_d\left(\int|\nabla\rho(\mathbf{x},t)|^2\rho(\mathbf{x},t)^{\frac{d-4}{2}}d\mathbf{x}\right)\left(\int\rho^{\frac{d}{2}}(\mathbf{x},t)d\mathbf{x}\right)^{2/d}.$$

We get the above bound by noting that (3) can be written as

$$\frac{d}{dt}\int\psi(\mathbf{x},t)^2 d\mathbf{x} = -4\frac{p-1}{p}\int|\nabla\psi(\mathbf{x},t)|^2 d\mathbf{x} + (p-1)\int\psi(\mathbf{x},t)^{\frac{2p+2}{p}}d\mathbf{x}$$

for

$$\psi(\mathbf{x},t) = \rho^{\frac{p}{2}}(\mathbf{x},t)$$

and using the inequality

$$\int|f(\mathbf{x})|^{\frac{2d+4}{d}}d\mathbf{x} \le C_d\left(\int|\nabla f(\mathbf{x})|^2 d\mathbf{x}\right)\left(\int|f(\mathbf{x})|^2 d\mathbf{x}\right)^{\frac{2}{d}}.$$

This inequality follows from the Sobolev inequality (*cf.* Evans and Gariepy [12])

$$\left(\int |f(\mathbf{x})|^{\frac{2d}{d-2}} d\mathbf{x}\right)^{\frac{d-2}{d}} \leq C_d \left(\int |\nabla f(\mathbf{x})|^2 d\mathbf{x}\right)^{\frac{1}{2}}$$

by straightforward interpolation (or Hölder inequality). A consequence of this is the fact that if the initial $\mathbf{L}^{\frac{d}{2}}$ norm of ρ is small then it is non-increasing in time; higher norms can be shown to decay also. This proves the regularity of solutions for initial data whose $\mathbf{L}^{\frac{d}{2}}$ norm is sufficiently small.

2.1. The Energy in unbounded domains

Consider the quantity

$$\mathcal{E}(\rho) = \int \left((\rho(\mathbf{x}) \log(\rho(\mathbf{x})) - \rho(\mathbf{x})) - \frac{1}{2}\rho(\mathbf{x}) c(\mathbf{x}) \right) d\mathbf{x}. \tag{4}$$

Note that in more than two dimensions the integral of ρc is positive and always

$$\int \rho c d\mathbf{x} = \int \int E(\mathbf{x}-\mathbf{y})\rho(\mathbf{x})\rho(\mathbf{y}) d\mathbf{x} d\mathbf{y}$$

The quantity $\mathcal{E}(\rho(\cdot, t))$ decays on solutions

$$\frac{d}{dt} \int \left(\rho(\log \rho - 1) - \frac{1}{2}\rho c \right) d\mathbf{x} = -\int \rho |\nabla(\log \rho - c)|^2 d\mathbf{x} = -\int \frac{\Gamma^2}{\rho} d\mathbf{x}.$$

Therefore, the energy dissipation vanishes only if the mass flux $\Gamma = 0$, so that ρ is a steady (time -independent) solution. Consequently, the system of equations in (1) do not have any periodic solutions with $\mathcal{E}$ finite, and the only possible asymptotic states of the system in (1) with a finite energy are steady (time-independent) solutions. Another possibility for the solutions is to have the energy diverge to $-\infty$, so that the density ρ diverges.

Note that, the functional derivative of $\mathcal{E}$ is

$$\frac{\delta \mathcal{E}}{\delta \rho} = \log \rho - c$$

so the equation takes the form

$$\partial_t \rho = \nabla \cdot \left(\rho \nabla \frac{\delta \mathcal{E}}{\delta \rho} \right) \tag{5}$$

The energy can be used to deduce information about steady states. If the energy and energy dissipation are well defined then the latter must vanish for time independent solutions. In the whole space the implications are the following: if ρ is the density of a smooth steady state and if it is strictly positive in an unbounded open connected set then it follows that $\log \rho - c$ is a constant in the same set. This implies that $\rho = k \exp(c)$ and because c vanishes at infinity it follows that ρ does not. This is impossible if the

energy is finite, so the assumptions are inconsistent: there are no steady states with a finite energy on an unbounded domain.

In this section, we have considered the definition of the energy $\mathcal{E}$ on unbounded domains. We can extend the definition of the energy for bounded domains with no-flux boundary conditions at the boundaries if we assume spherical symmetry and we present the details below in Sec. 2.3.

2.2. Radial Equations

We will henceforth restrict ourselves to solutions with spherical symmetry and treat the space dimension d as a continuous parameter in the equation so that it is no longer restricted to assume only integer values.

Let $m(r,t)$ be given by

$$m(r,t) = \int_0^r \rho(r,t) r^{d-1} dr, \tag{6}$$

so that $m(r)$ is the mass inside a sphere of radius r divided by $C_d = d\pi^{d/2}/\Gamma((d+2)/2)$ (the "surface area" of the unit sphere in d dimensions). From the system in Eq. (1), we can deduce that $m(r,t)$ satisfies

$$\partial_t m = \partial_r^2 m - \frac{d-1}{r}\partial_r m + \rho m. \tag{7}$$

Under a scale transformation $r \to r/L, t \to t/L^2$, the transformation of the quantity m is given by

$$m_L(r,t) = L^{d-2} m\left(\frac{r}{L}, \frac{t}{L^2}\right)$$

In the variable ρ, we have the integro-differential equation

$$\partial_t \rho = \partial_r^2 \rho + \frac{d-1}{r}\partial_r \rho + \frac{1}{r^{d-1}}\partial_r(m\rho). \tag{8}$$

The boundary conditions are ρ decaying at infinity for unbounded domains and zero mass flux at the boundaries for bounded domains.

All the numerical results we present in this paper are for solutions on bounded domains. We use the form of the equations in Eq. (8) along with the definition of m in (6) and the no-flux boundary conditions for our numerical simulations. All the simulations were done using a mass conserving, second-order in time step, implicit scheme along with adaptive mesh refinement. Some of the numerical results reported in this paper were obtained with numerical routines that were developed using BuGS [13], a general purpose toolkit for solving parabolic PDEs.

In our analysis, we will also sometimes work with the coordinate $\chi = r^d/d$. In this coordinate, we have

$$\rho = \partial_\chi m, \tag{9}$$

and Eq. (7) transforms into

$$\partial_t m(\chi) = K_d \chi^\alpha \partial_\chi^2 m + m\partial_\chi m. \tag{10}$$

where $\alpha = 2 - 2/d$ and $K_d = d^\alpha$. We define the quantities $b(r)$ and $h(r)$ by

$$b(r) = \frac{m(r)}{\chi} = \frac{dm(r)}{r^d}, \tag{11}$$

and

$$h(r) = \frac{m(r)}{2r^{d-2}}. \tag{12}$$

Under a scale transformation, b transforms like ρ *i.e.*,

$$b_L(r,t) = \frac{1}{L^2} b\left(\frac{r}{L}, \frac{t}{L^2}\right),$$

and h transforms as

$$h_L(r,t) = h\left(\frac{r}{L}, \frac{t}{L^2}\right).$$

so that h does not pick up any powers of L under a scale transformation. The variables b and h turn out to be the most convenient variables to analyze this system. $b(r,t)$ satisfies the equation

$$\begin{aligned} \partial_t b(r,t) &= \Delta_{d+2} b(r,t) + \rho b(r,t) \\ &= \partial_r^2 b(r,t) + \left(\frac{d+1}{r} + \frac{r b(r,t)}{d}\right) \partial_r b(r,t) + b^2(r,t), \end{aligned} \tag{13}$$

where

$$\Delta_{d+2} = \partial_r^2 + \frac{d+1}{r}\partial_r$$

is the $d+2$ dimensional radial Laplacian. $h(r,t)$ satisfies the equation

$$\partial_t h(r,t) = \partial_r^2 h(r,t) + \left(\frac{d-3+2h(r,t)}{r}\right) \partial_r h(r,t) + \frac{2(d-2)h(r,t)}{r^2}(h(r,t)-1) \tag{14}$$

2.3. Energy in a bounded domain

In this section we will work with the Eq. (10) for $\chi \in [0,1]$, with $1 < \alpha < 2$ ($2 < d < \infty$) and boundary conditions $m(0) = 0$, $m(1) = M$. The positive density $\rho = m_\chi$ solves

$$\rho_t = K_d \partial_\chi \left[\chi^\alpha \rho \partial_\chi \left(\log \rho - c\right)\right] \tag{15}$$

where $K_d \chi^\alpha c_\chi = -m$. The flux $K_d \chi^\alpha \rho_\chi + m\rho$ of (15) vanishes at the boundaries (10). In terms of ρ, c is given by

$$c(\chi,t) = \int_0^1 k(\chi,z)\rho(z,t)dz + A(t) \tag{16}$$

with the kernel $k(\chi,z) = 0$ if $z \geq \chi$ and

$$k(\chi,z) = -\frac{1}{K_d}\int_z^\chi y^{-\alpha} dy$$

if $z \leq \chi$. The constant of integration $A(t)$ is not specified. We write

$$k(\chi,z) = s(\chi,z) + j(\chi,z)$$

where

$$s(\chi,z)=\frac{1}{2}\left(k(\chi,z)+k(z,\chi)\right)$$

and

$$j(\chi,z)=\frac{1}{2}\left(k(\chi,z)-k(z,\chi)\right)$$

are the symmetric and anti-symmetric parts of k. We denote

$$(S\rho)(\chi,t)=\int_0^1 s(\chi,z)\rho(z,t)dz$$

and

$$(J\rho)(\chi,t)=\int_0^1 j(\chi,z)\rho(z,t)dz.$$

An explicit calculation yields

$$(J\rho)(\chi,t)=\int_0^1 j(\chi,z)\rho(z,t)dz=B(t)+g(\chi)$$

with $B(t)$ independent of χ:

$$B(t)=-\frac{1}{2K_d(\alpha-1)}\int_0^1\frac{\rho(z,t)}{z^{\alpha-1}}dz$$

and $g(\chi)$ independent of t:

$$g(\chi)=\frac{M}{2K_d(\alpha-1)\chi^{\alpha-1}}.$$

We define $C(t)$ to be

$$C(t)=A(t)-B(t).$$

With this notation

$$c(\chi,t)=\int_0^1 s(\chi,z)\rho(z,t)dz+C(t)+\frac{M}{2K_d(\alpha-1)\chi^{\alpha-1}}=S\rho+C+g \quad (17)$$

with a symmetric kernel s (i.e., $s(\chi,z)=s(z,\chi)$). Therefore

$$\frac{1}{2}\frac{d}{dt}\int_0^1\rho(\chi,t)c(\chi,t)d\chi=\int_0^1\rho_t(\chi,t)\left((S\rho)(\chi,t)+\frac{1}{2}(C(t)+g(\chi))\right)d\chi+\frac{M}{2}\frac{dC(t)}{dt}$$

and thus

$$\frac{d}{dt}\left[\int_0^1\frac{1}{2}\rho(\chi,t)\left(c(\chi,t)+C(t)+g(\chi)\right)d\chi-MC(t)\right]=\int_0^1\rho_t(\chi,t)c(\chi,t)d\chi.$$

Absorbing the $MC(t)$ term in the integral, in the bounded domain case the energy is defined to be

$$\mathcal{E}(\rho)=\int_0^1\rho(\chi)(\log(\rho(\chi))-1)-\frac{1}{2}\rho(\chi)\left(c(\chi)-C(t)+\frac{M}{2K_d(\alpha-1)\chi^{\alpha-1}}\right)d\chi. \quad (18)$$

On solutions of (10) we have

$$\frac{d}{dt}\mathcal{E}=\int_0^1\rho_t(\chi,t)\left(\log\rho(\chi,t)-c(\chi,t)\right)d\chi\le 0.$$

2.4. Separation properties of solutions

Let $m_S(r,t)$ be a solution to Eqs.(6) and (7). Then, if $m(r,t)$ is another solution to the equations, we have the following comparison principle:

If $m(r,t_0) \leq m_S(r,t_0)$ holds at some time t_0 *for all* r then

$$m(r,t) \leq m_S(r,t)$$

for all $t \geq t_0$. The proof of this fact is done by passing to the variable g defined by

$$m_S(r,t)g(r,t) = m(r,t)$$

and noting that it obeys an equation of the form

$$g_t = \partial_r^2 g + V(r,t)\partial_r g + \rho_S(r,t)(g(g-1))$$

where

$$\rho_S(r,t) = \frac{1}{r^{d-1}}\partial_r m_S(r,t),$$

is the density corresponding to the solution $m_S(r,t)$ and is non-negative for all r and t. The velocity $V(r,t)$ is:

$$V(r,t) = 2\partial_r \log(m_S) - \frac{d-1}{r} + \frac{m_S g}{r^{d-1}}.$$

This equation has a maximum principle so that if $0 \leq g \leq 1$ initially, then this inequality is preserved in time.

The comparison principle can be equivalently restated as:

If $b(r,t_0) \leq b_S(r,t_0)$ holds at some time t_0 for all r, then $b(r,t) \leq b_S(r,t)$ for all $t \geq t_0$ and similarly if $h(r,t_0) \leq h_S(r,t_0)$ at some time t_0 for all r, then $h(r,t) \leq h_S(r,t)$ for all $t \geq t_0$.

3. Time independent solutions

We will now investigate the steady (time-independent) solutions of Eq. (7). The steady solutions $m_S(r)$ are given by

$$r^{d-1}\partial_r\left(\frac{\partial_r m_S}{r^{d-1}}\right) + \rho_S m_S = 0 \tag{19}$$

Using Eq. (6), the above equation can be written as the following system of coupled ODEs

$$\begin{aligned} \partial_r \rho_S &= -\frac{\rho_S m_S}{r^{d-1}}, \\ \partial_r m_S &= \rho_S r^{d-1}. \end{aligned} \tag{20}$$

The steady states are solutions to the ODEs with the initial conditions

$$\rho_S(0) = \rho_0, \lim_{r\to 0}\frac{dm_S(r)}{r^d} = \rho_0 \tag{21}$$

From Eq. (20), we have

$$\rho_S(r) = \rho_S(0) \exp\left(- \int_0^r \frac{m_S(r')}{r'^{d-1}} dr' \right)$$

and using the fact that $m(r) \to \rho_S(0) r^d/d$ as $r \to 0$, we have that $\rho_S(r) > 0$ as long as

$$\left| \int_0^r \frac{m_S(r')}{r'^{d-1}} dr' \right| < \infty.$$

Eq. (20) with $\rho_S(r') > 0$ for $0 \le r' \le r$ implies that $m(r) \ge 0$. The above equation implies that $\rho_S(r) \le \rho_S(0)$ and it follows that $m_S(r) \le \rho_S(0) r^d/d$. This implies that there exist global solutions to Eq. (20) with the initial conditions in (21). By the uniqueness of the solutions to these ODEs, we have a one-parameter family of steady solutions parameterized by the density at the origin ρ_0. These solutions are related to each other by the scale transformation

$$r \to r/L, \rho_L(r) = L^{-2} \rho(r/L)$$

Consequently, we can obtain all the steady solutions by integrating the system of equations in (20) with $\rho_0 = 1$ and rescaling appropriately.

3.1. Steady solutions in $d < 2$

For $d < 2$, the balance of the dominant terms in Eq. 19 for large r is given by

$$\partial_r^2 m_S + \frac{m_S \partial_r m_S}{r^{d-1}} = 0 \tag{22}$$

and the solutions to this equation are asymptotically given by

$$m_S(r) = M[1 - a r^{d-1} \exp(-M/r^{d-2})], \tag{23}$$

where M and a are constants determined by the initial conditions. We have

$$M = \lim_{r \to \infty} m_S(r)$$

Consequently, the steady state solutions for $d < 2$ have a finite mass even for unbounded domains.

Fig. 3 shows plot of m_S as a function of r for $d = 1.5$. The densities at the origin for the two curves are $\rho_0 = 1$ and $\rho_0 = 10$. Note that m_S asymptotes to a finite value for large r.

Under a scale transformation, we have

$$r_L = r/L, M_L = L^{d-2} M, (\rho_0)_L = L^{-2} \rho_0$$

Given any initial mass distribution $m(r, t_0)$, by making the scale factor L sufficiently small, we can find a steady solution m_S such that $m(r, t_0) \le m_S(r)$. Consequently, the comparison principle will imply that $m(r, t) \le m_S(r)$ for all $t \ge t_0$. This suggests that

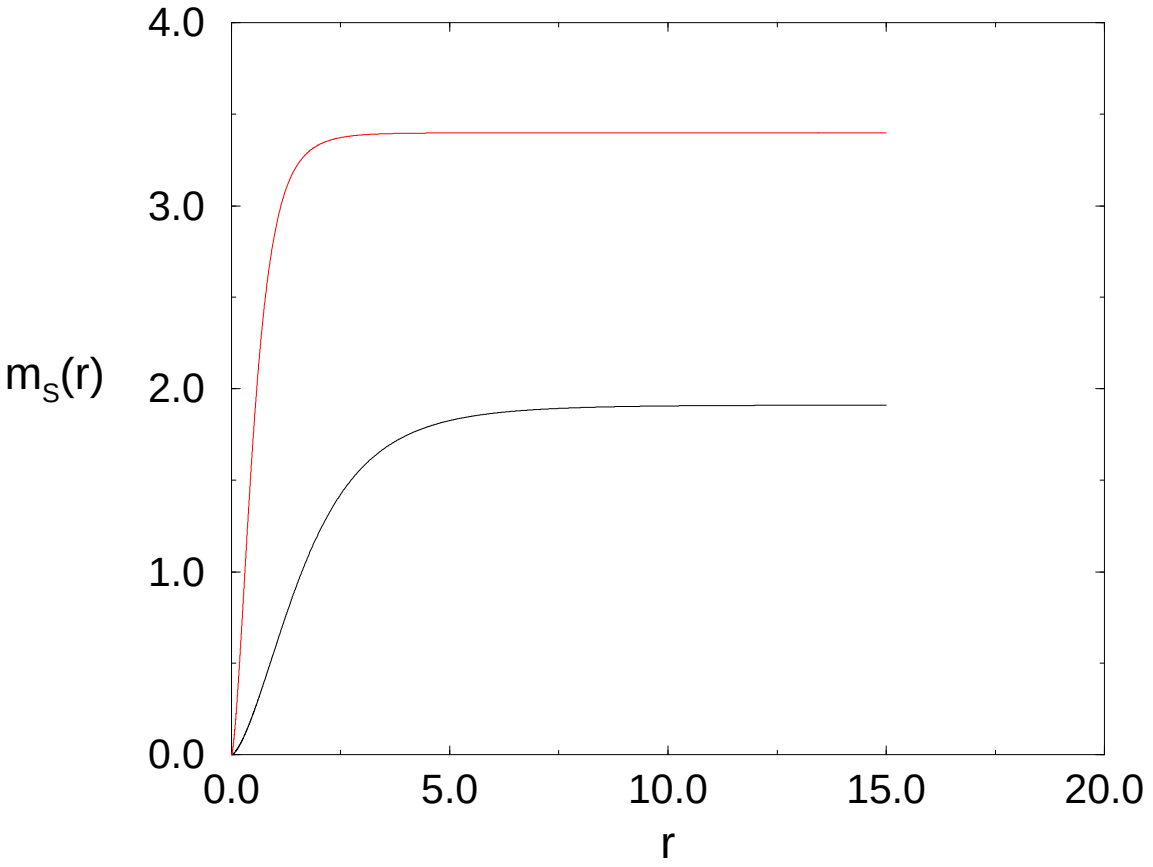

Figure 3. Steady solutions in $d = 1.5$

the solutions $m(r,t)$ for $d < 2$ are regular for all t and we will show that this is indeed the case.

We can also deduce the properties of steady solutions in bounded domains using the steady solution with $\rho_0 = 1$ and scale transformations. Let Ω_R be the bounded domain $r \leq R$. Then, we have

$$m_L(R) = L^{d-2} m_S(R/L). \tag{24}$$

$m_S(r)$ is a continuous, monotone increasing function of r and for $d < 2$, L^{d-2} and R/L are continuous monotone decreasing functions of L. Therefore, $m_L(R)$ is a continuous monotone decreasing function of L. Also,

$$m_L(R)|_{L\to 0} \to \infty \text{ and } \; m_L(R)|_{L\to\infty} \to 0.$$

This implies that for any given $M_0 \in (0,\infty)$, there exists a unique L_0 such that $m_{L_0}(R) = M_0$. Consequently, there exists a unique steady solution with a given total mass in a bounded domain of size R.

3.2. Steady solutions in $d = 2$

For $d = 2$, Eq. (19) becomes

$$\partial_r^2 m_S + \frac{m_S - 1}{r}\partial_r m_S = 0 \tag{25}$$

The solutions to this equation are given by

$$m_S(r) = \frac{4r^2}{k + r^2}, \tag{26}$$

where $k > 0$ is a constant determined by the density at the origin. We have

$$M = \lim_{r\to\infty} m_S(r) = 4$$

Consequently, the steady state solutions for $d = 2$ have a finite mass even for unbounded domains and the asymptotic value of m_S is 4, independent of the density at the origin.

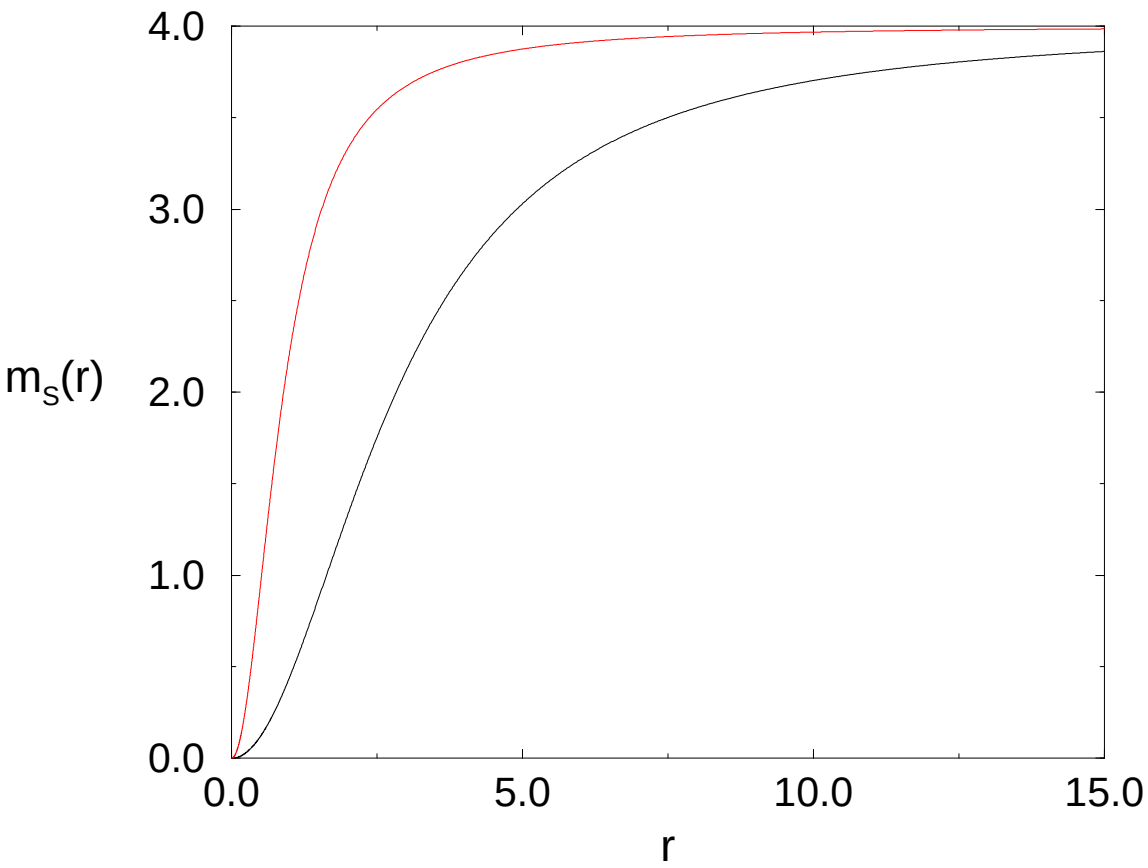

Figure 4. Steady solutions for $d = 2$.

Fig. 4 shows plot of m_S as a function of r for $d = 2$. The densities at the origin for the two curves are $\rho_0 = 1$ and $\rho_0 = 10$. Note that $m_S \sim 4$ for large r.

Under a scale transformation, we have

$$r_L = r/L, M_L = M = 4, (\rho_0)_L = L^{-2}\rho_0, k_L = L^{-2}k$$

Given any initial mass distribution $m(r, t_0)$, with $m(r, t_0) \leq 4$ *for all* r, by making the scale factor L sufficiently small, we can find a steady solution m_S such that $m(r, t_0) \leq m_S(r)$. Consequently, the comparison principle will imply that $m(r, t) \leq m_S(r)$ for all $t \geq t_0$. This suggests that the solutions $m(r, t)$ for $d = 2$ are regular for all t if $m(r, t_0) \leq 4$ for all r.

In a bounded domain $\Omega_R = \{x \in R^2 : ||x|| \leq R\}$, we have

$$M = m(R) = \frac{4R^2}{k + R^2} < M_c = 4 \tag{27}$$

Therefore, for any given mass less than $C_2 M_c = 8\pi$, there exists a unique steady state solution on the bounded domain Ω_R with that mass ($C_2 = 2\pi$ is the circumference of the unit circle). Also, there are no steady solutions with a total mass bigger than 8π on a bounded domain. It is known that all initial conditions with a total mass bigger than 8π will lead to a blowup of the density in a finite time [14, 10].

3.3. Steady solutions in $d > 2$

For $d > 2$, Eq. 19 becomes

$$\partial_r^2 m_S + \left(\frac{m_S}{r^{d-1}} - \frac{d-1}{r} \right) \partial_r m_S = 0 \tag{28}$$

and the solutions to this equation are asymptotically given by

$$m_S(r) = 2r^{d-2}, \tag{29}$$

independent of the initial density ρ_0. Therefore, there are no finite mass steady state solutions for $d > 2$ on an unbounded domain (Also see Sec. 2.1). There, however are finite mass steady solutions on bounded domains. We will restrict our attention to bounded domains in the rest of this section.

Fig. 5 shows plot of m_S as a function of r for $d = 3$. The densities at the origin for the two curves are $\rho_0 = 1$ and $\rho_0 = 10$. Note that both the curves have the same asymptotic behavior. Also note that the two curves intersect, implying that the steady solutions with a given mass in a box of a given size are not necessarily unique. We will return to this point below.

Theorem 1. *For $2 < d$, we have*

$$h_S(r) = \frac{m_S(r)}{2r^{d-2}} \to 1 \text{ as } r \to \infty.$$

Further, for $2 < d < 10$, the solution $h_S(r)$ has a countable number of oscillations about the asymptotic value of $h = 1$ and for $d > 10$, the solution is monotonic on $(0, \infty)$.

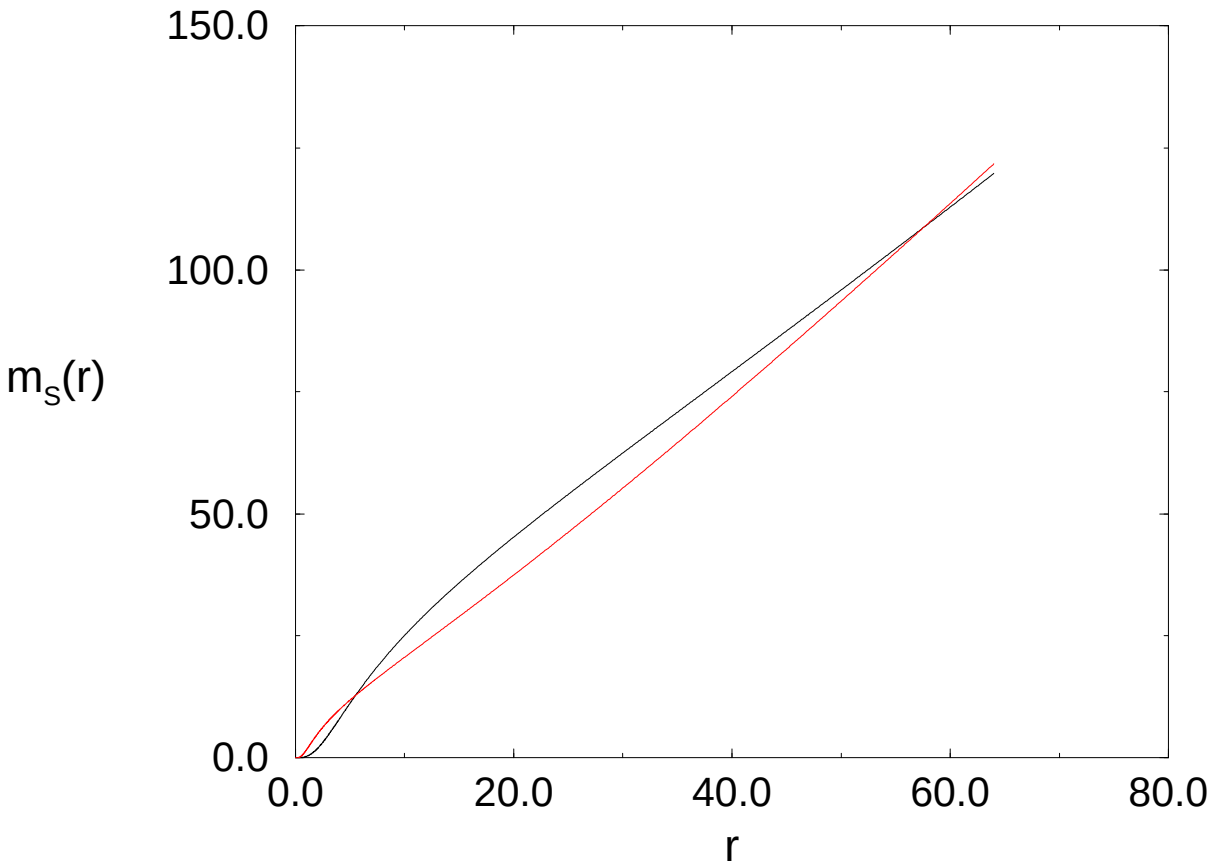

Figure 5. Steady solutions for $d = 3$.

Proof. The equation for h has a scale invariance $h_L \to h$ and $r_L \to r/L$ *i.e*, the equations are invariant under the above transformation. Therefore, if we use the variable $z = \log(r)$ instead of r, the scale invariance implies that the equations are invariant under the transformation $h \to h$, $z \to z + \zeta$ for any ζ. Therefore, in terms of z, we will have an autonomous dynamical system :

$$\begin{aligned} \frac{dh}{dz} &= g - (d-2)h \\ \frac{dg}{dz} &= -2(h-1)g. \end{aligned} \tag{30}$$

where $g = (d-2)h + dh/dz = r^2\rho/2$. The only critical points of the system in (30) are $(0,0)$ and $(1,(d-2))$. Also, the line $g = 0$ is invariant under the flow.

Linearizing about $(0,0)$, we have

$$\begin{pmatrix} h' \\ g' \end{pmatrix} = \begin{pmatrix} -(d-2) & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} h \\ g \end{pmatrix} + O(h^2+g^2).$$

Therefore, the fixed point at $(0,0)$ is a saddle. The stable direction is given by $g=0$ and the unstable direction is given by $g=dh$.

Linearizing about the point $(1,(d-2))$, we have

$$\begin{pmatrix} \delta h' \\ \delta g' \end{pmatrix} = \begin{pmatrix} -(d-2) & 1 \\ -2(d-2) & 0 \end{pmatrix} \begin{pmatrix} \delta h \\ \delta g \end{pmatrix} + O(\delta h^2+\delta g^2).$$

where $\delta h = h-1$ and $\delta g = g-(d-2)$. Computing the eigenvalues, we see that the fixed point at $(1,(d-2))$ is a stable focus for $2<d<10$ and is a node for $d>10$.

We are interested in solutions satisfying the initial condition $\rho \to \rho_0, h \to \rho_0 r^2/2d$ as $r \to 0$. Therefore, the solution (orbit) should approach $(0,0)$ as $z \to -\infty$, so that we are looking for the unstable manifold of the saddle at $(0,0)$.

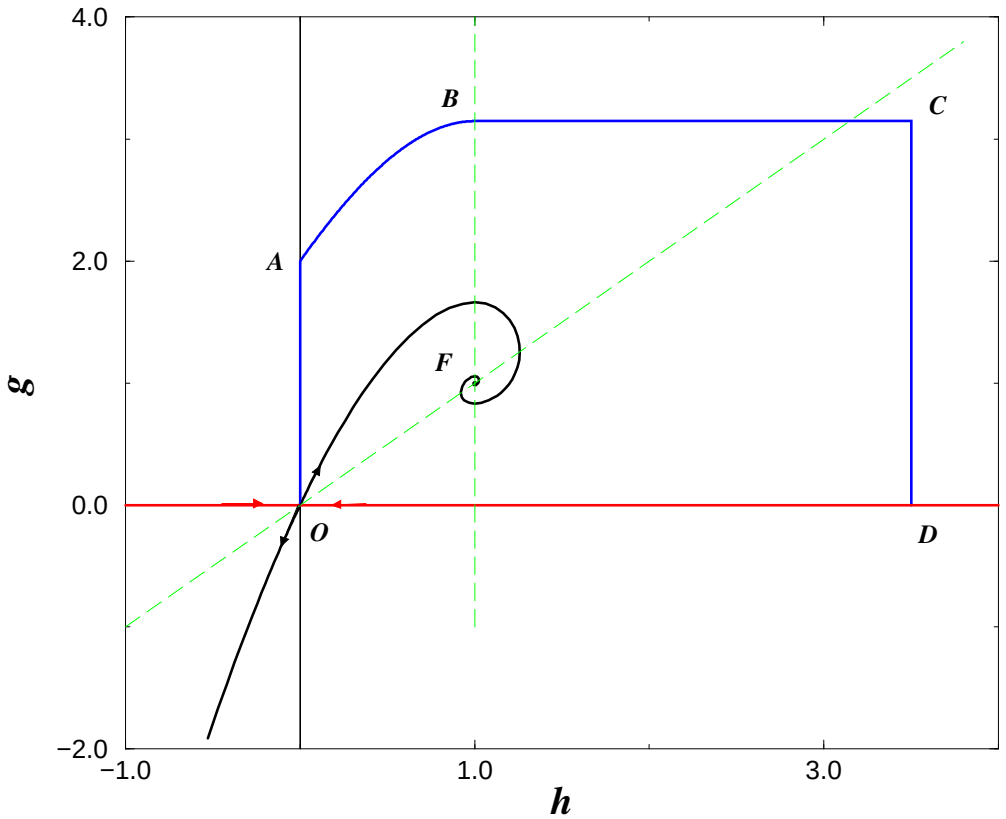

Figure 6. Portrait of the Phase plane of the system in Eq. (30) with $d=3$.

Figure 6 shows the phase plane for the system in $d = 3$. We will now show that the basin of attraction of the stable focus(node) at $(1,(d-2))$ includes the open set bounded by $OABCD$ and therefore, one branch of the unstable manifold of the saddle connects the saddle with the stable focus(node).

The point A is chosen to be $(0, d-1)$ and the segment AB is generated by following the flow in Eq. (30) starting at A ($z = 0$) and intersecting the line $h = 1$ at B ($z = z_0$). In the region $h \le 1, g \ge 0$, we have $g' \ge 0$ so that for the portion AB, $g(z) \ge (d-1)$. Consequently, $h' \ge 1$ so that $z_0 \le 1$. Also, $g' \le 2g$ for this portion of the curve so that at B, we have $g \le g(0)e^{2z_0} \le (d-1)e^2$.

Let the value of g at B be g_0. The segment BC is chosen so that $g = g_0$ and $1 \le h \le g_0/(d-2) + 1$. On this segment, we have $g' \le 0$. The segment CD is given by $h = g_0/(d-2) + 1$, $0 \le g \le g_0$. On this segment, $h' \le 0$. The segment OD is a part of the stable manifold of O. Finally, on the segment OA, we have $h' \ge 0$. Therefore, we see that an orbit that starts from the interior of the set bounded by $OABCD$ cannot leave the set.

Let $S(y)$ be the set obtained by following the orbits of all the initial conditions in a given set S from $z = 0$ to $z = y$. The line $g = 0$ is invariant (the stable manifold of $(0,0)$) so that if $S \subset \Sigma = \{(h,g) \,|\, g > 0\}$, we have that $S(y) \subset \Sigma \quad \forall y \ge 0$. We define the quantity $A(y)$ by

$$A(y) = \int_{S(y)} \frac{1}{g}\, dhdg.$$

From Eq. (30) we get that

$$\frac{d}{dy}A(y) = -(d-2)A(y),$$

which implies that there are no invariant sets with $A(y)$ finite but nonzero.

Consequently, there are no limit cycles in the interior of the set $OABCD$ and this implies that the unstable manifold of the saddle at $(0,0)$ connects to the stable focus(node) at $(1,(d-2))$.

From the above result, we obtain that, for large r,

$$\begin{aligned} h_S(r) &\sim 1 + a_1 r^{-d/2+1} \cos(\beta \log(r/r_0)) \qquad \text{for } 2 < d < 10, \\ h_S(r) &\sim 1 + a_2 r^{-d/2+1+\gamma} \qquad \text{for } d > 10, \end{aligned} \tag{31}$$

where

$$\beta = \frac{\sqrt{(d-2)(10-d)}}{2}, \qquad \gamma = \frac{\sqrt{(d-2)(d-10)}}{2}$$

and, a_1, a_2 and r_0 are constants that depend on the density at the origin ρ_0. □

Fig. 7 shows a plot of h_S as a function of r for $d = 3$. The density at the origin is $\rho_0 = 20$. Note that the curve oscillates about the asymptotic value of 1 and that the period of the oscillations grows with r.

Using the result in Theorem 1 along with the scale invariance of the solutions, we can deduce the properties of the steady solutions in bounded domains.

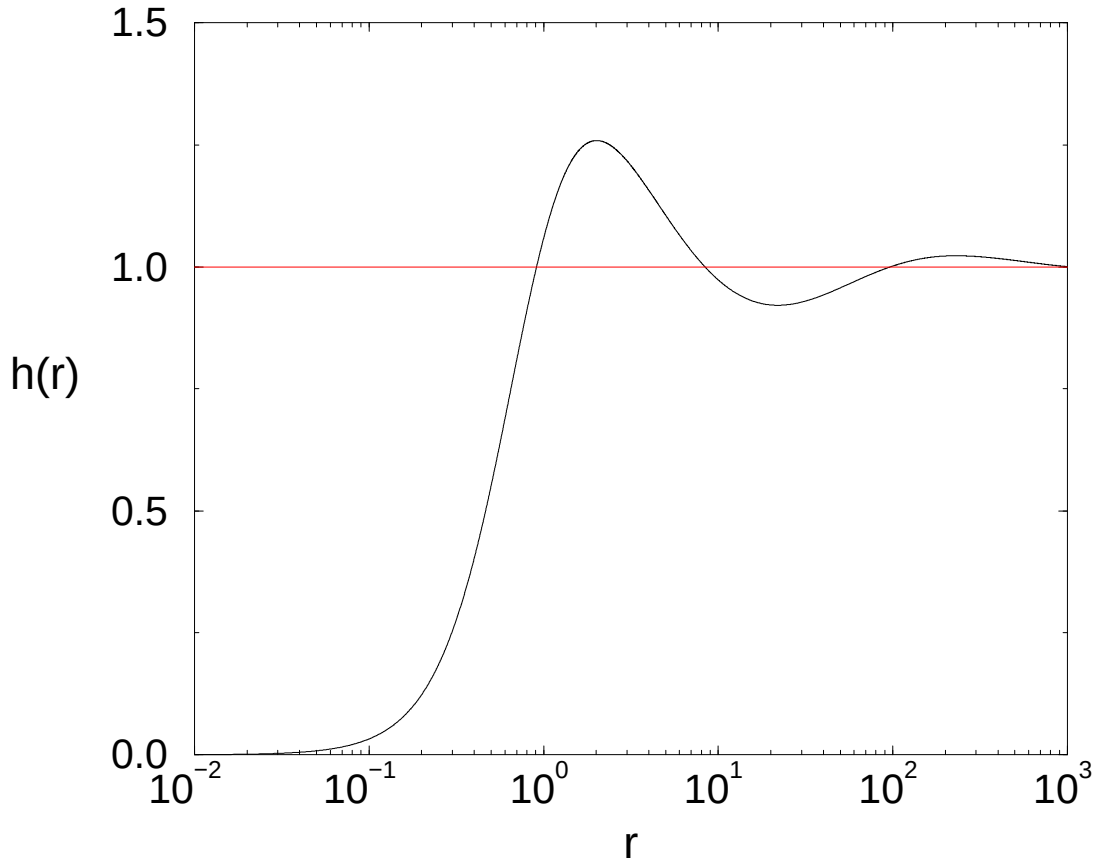

Figure 7. h_S as a function of r for $d = 3$. $\rho_0 = 20$

We will now investigate the conditions under which there exist steady solutions in a bounded domain Ω_R in d dimensions with a given total mass $C_d M$ (C_d as defined above is the "area" of a unit sphere in d dimensions). We define the quantity

$$\mu = h(R) = \frac{M}{2R^{d-2}}$$

Under a scale transformation, we have

$$r_L = r_1/L, h_L(r) = h_1(r/L)$$

Since all the steady solutions can be obtained by appropriate scale transformations of a particular steady solution (say the solution $h_S(r)$), it follows that there exists a steady solution in Ω_R with a total mass $C_d M$ if and only if we can find a r_a such that

$$h_S(r_a) = \mu \tag{32}$$

For $2 < d < 10$, let μ_i represent the ith extremum of the curve $h_S(r)$ with $r \in (0, \infty)$, *i.e.* not counting the point $(0, 0)$ and let $\mu_0 = 0$. Since the point $(1, d-2)$ is a stable

focus for the system in (30) and $dh/dr = 0$ only on the line $g = (d-2)h$, we have that

$$\mu_1 > \mu_3 > \mu_5 > \ldots > 1,$$

and

$$0 = \mu_0 < \mu_2 < \mu_4 < \mu_6 < \ldots < 1.$$

(See Fig. 7 for an illustration of these inequalities for $d = 3$). Consequently, there are no solutions to Eq. (32) if $\mu > \mu_1$. There is one solution for $\mu_0 = 0 < \mu < \mu_2$, two solutions for $\mu_3 < \mu < \mu_1$ and so on.

Let r_i denote the $i+1$th solution to Eq. (32), *i.e*, r_i is in between $h_S^{-1}(\mu_i)$ and $h_S^{-1}(\mu_{i+1})$. The scale factor L for obtaining the solution in the bounded domain Ω_R is given by

$$L_i = R/r_i.$$

Therefore, for every solution r_i of Eq. (32), we can find a steady solution in Ω_R given by

$$h_i(r) = h_S\left(\frac{r_i}{R}r\right)$$

Note that $h_i(r)$ has i extrema in $(0, R)$ *i.e.*, there are i points in the interior of the domain where $\partial_r h_i(r) = 0$.

If $M > M_c$ with

$$M_c = 2\mu_1 R^{d-2}, \tag{33}$$

Eq. (32) has no solutions. Consequently, there exists a critical mass M_c such that there exist no spherically symmetric steady solutions in a bounded domain Ω_R with a mass larger than $C_d M_c$.

Also, a steady solution, if it exists, is not necessarily unique in contrast to the situations in $d \leq 2$. Fig. (8) shows three different steady solutions with the same mass $4\pi M = 80\pi$ in a bounded domain of radius 10.

For $d > 10$, the $h_S(r)$ increases monotonically from 0 to 1. Therefore, there exist no steady solutions with mass larger than $C_d M_c = 2C_d R^{d-2}$ for a space dimension d. In this case however, if a steady solution exists with a given mass, it is unique.

3.4. Stability of steady solutions in a bounded domain

As we saw above, we can have more than one steady solution with a given total mass and a given bounded domain Ω_R in dimensions $2 < d < 10$. We will now prove the following result concerning the stability of the various steady solutions.

Theorem 2. *: Let $\mu = M/(2R^{d-2})$ be such that there exist p steady solutions $h_0(r), h_1(r), \ldots, h_{p-1}(r)$ with the given total mass on the bounded domain Ω_R. Then, the solution $h_i(r)$ that has i extrema for $r \in (0, R)$ with $i \geq 1$ is unstable and the corresponding unstable manifold has a dimension i.*

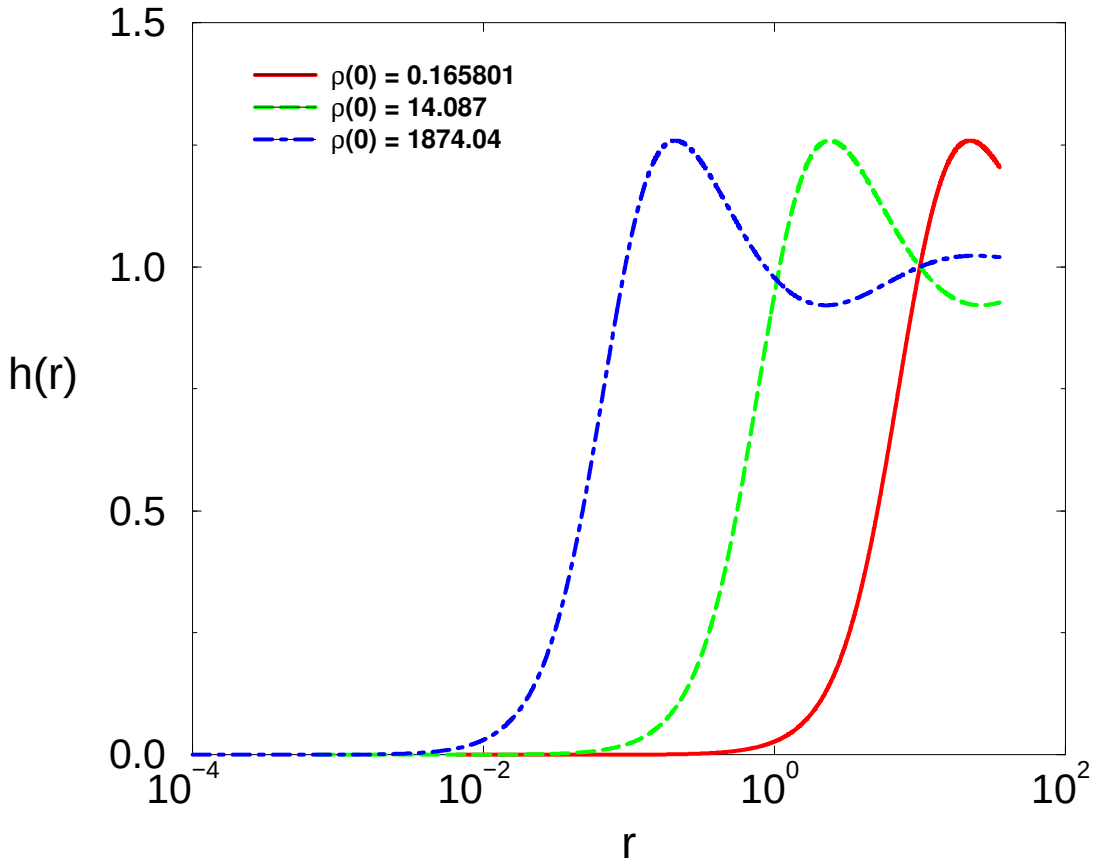

Figure 8. Steady solutions with $\mu = 1$.

Proof. Linearizing Eq. (14) about $h = h_i$, we obtain

$$\partial_t \phi(r,t) = \partial_r^2 \phi(r,t) + \left(\frac{d-3+2h_i(r,t)}{r}\right)\partial_r \phi(r,t) + \frac{2r\partial_r h_i(r) + 2(d-2)(2h_i(r)-1)}{r^2}\phi(r,t) \tag{34}$$

We will consider the eigenvalue problem for the above equation. let ϕ_λ be a solution of

$$\partial_r^2 \phi_\lambda(r) + \left(\frac{d-3+2h_i(r,t)}{r}\right)\partial_r \phi_\lambda(r) + \left[\frac{2r\partial_r h_i(r) + 2(d-2)(2h_i(r)-1)}{r^2} + \lambda_n\right]\phi_\lambda(r) = 0 \tag{35}$$

with the initial conditions

$$\phi_\lambda(r) \to r^2 \text{ as } r \to 0.$$

We are interested in finding λ_n so that the solution $\phi_{\lambda_n}(r)$ satisfies $\phi_{\lambda_n}(R) = 0$. If we

define $\psi_\lambda(r) = \phi_\lambda(r)v(r)$, where $v(r)$ is given by

$$v(r) = \exp\left[\frac{1}{2}\int_1^r \frac{d-3+2h_i(r')}{r'}\,dr'\right],$$

$\psi_\lambda(r)$ satisfies the equation

$$\partial_r^2\psi_\lambda(r) + (q(r)+\lambda)\psi_\lambda(r) = 0$$

with

$$q(r) = \left[\frac{4r\partial_r h_i(r) + 8(d-2)(2h_i(r)-1) + 1 - (d-2+2h_i(r))^2}{r^2}\right].$$

Off the origin, $v(r) > 0$. Therefore, the zeroes of ψ_λ and ϕ_λ coincide off the origin. Also, if $\lambda_m > \lambda_n$, by the Sturm comparison theorem, ψ_{λ_m} vanishes between every two zeroes of ψ_{λ_n} and therefore, it is also true that ϕ_{λ_m} vanishes between every two zeroes of ϕ_{λ_n}. It is easily seen that the locations of the zeroes of ϕ_λ are continuous functions of λ.

The steady solutions of Eq. (14) are scale invariant, so that $h_i(r/L)$ is a steady solution for all L. Consequently, $h_i(r/L)$ satisfies

$$\begin{aligned}&\partial_r^2 h_i(r/L) + \left(\frac{d-3+2h_i(r/L)}{r}\right)\partial_r h_i(r/L)\\ &+\frac{2(d-2)h_i(r/L)(h_i(r/L)-1)}{r^2} \qquad\qquad = 0\end{aligned} \tag{36}$$

Differentiating the above equation *w.r.t.* L and setting $L = 1$, we obtain

$$\begin{aligned}&\partial_r^2 f_0(r) + \left(\frac{d-3+2h_i(r,t)}{r}\right)\partial_r f_0(r)\\ &+\left[\frac{2r\partial_r h_i(r) + 2(d-2)(2h_i(r)-1)}{r^2}\right] f_0(r) = 0,\end{aligned} \tag{37}$$

where

$$f_0(r) = r\partial_r h_i(r).$$

Comparing Eq. (37) with Eq. (35), we see that $f_0(r)$ is an eigenfunction with an eigenvalue $\lambda = 0$ if it satisfies the boundary condition $f_0(R) = 0$. The function $h_i(r)$ has i extrema in $(0, R)$. Therefore, the function $\phi_0(r)$ has i zeroes in the open set $(0, R)$. By the Sturm comparison theorem and the continuity of the location of the zeroes as a function of λ, there exists a $\lambda^*_{-1} < 0$ such that $\phi_{\lambda^*_{-1}}(R) = 0$ and there are exactly $i-1$ zeroes for $\phi_{\lambda^*_{-1}}(r)$ in $(0, R)$ and a $\lambda^*_1 \geq 0$ such that $\phi_{\lambda^*_1}(R) = 0$ and there are exactly i zeroes for $\phi_{\lambda^*_1}(r)$ in $(0, R)$.

Now, by standard results from Sturm-Liouville theory, it follows that there exist $\lambda^*_{-i} < \ldots < \lambda^*_{-2} < \lambda^*_{-1} < 0$ such that $\phi_{\lambda^*_{-k}}(R) = 0$ and $\phi_{\lambda^*_{-k}}(r)$ has exactly $i-k$ zeroes in $(0, R)$.

All the eigenfunctions with eigenvalues less than zero are unstable.This proves our claim that the steady solution $h_i(r)$ is unstable for $i \geq 1$ and the unstable manifold has dimension i. □

4. Properties of solutions

In the previous section, we analyzed the time-independent (steady) solutions of the KS system for different space dimensions d. In this section, we will analyze some of the properties of the time-dependent solutions. We begin by showing that there exist spherically symmetric solutions that blow up in a finite time for $d \geq 2$.

4.1. Blowup of radial solutions for $d \geq 2$

Theorem 3. *For $d \geq 2$, there exist solutions of the radial equations (7) such that the density blows up at a finite time,* i.e. *we can find sequences $r_n \to r^*$ and $t_n \to T^* < \infty$ such that $\limsup_{n\to\infty} \rho(r_n, t_n) = \infty$.*

Proof. For radial solutions, Eq. (13) gives

$$\partial_t b(r,t) = \Delta_{d+2} b(r,t) + \rho b(r,t) \tag{38}$$

where

$$\Delta_{d+2} = \partial_r^2 + \frac{d+1}{r}\partial_r \tag{39}$$

is the $d+2$ dimensional radial Laplacian. Now we take a function γ that depends on a free parameter T

$$\gamma(r,t) = c_{d+2}(T-t)^{-\frac{d+2}{2}} \exp\left(-\frac{r^2}{4(T-t)}\right). \tag{40}$$

This function solves the backward $d+2$ dimensional heat equation with final condition $\gamma(\cdot, T) = \delta$. The constant c_{d+2} is adjusted so that

$$\int_0^\infty \gamma(r,t) r^{d+1} dr = 1.$$

We note for later use that

$$\partial_r \gamma = -\frac{r}{2(T-t)}\gamma.$$

The constant T will be chosen later. Now we multiply the equation (38) by γ, integrate $r^{d+1}dr$ and use the properties of γ. After checking that there are no boundary contributions we obtain:

$$\frac{d}{dt}\int_0^\infty b(r,t)\gamma(r,t) r^{d+1} dr = \int_0^\infty \rho(r,t) b(r,t)\gamma(r,t) r^{d+1} dr.$$

We recall that

$$r^{d-1}\rho = \partial_r m$$

and that

$$r^d b(r,t)/d = m.$$

We deduce

$$\frac{d}{dt}\int_0^\infty b(r,t)\gamma(r,t) r^{d+1} dr = \frac{1}{2d}\int_0^\infty \left[\frac{r^2}{2(T-t)} + (d-2)\right] b^2(r,t)\gamma(r,t) r^{d+1} dr. \tag{41}$$

Now, if $d > 2$ we discard the first term in the square brackets and use the fact that

$$\left(\int b(r)\gamma(r) r^{d+1} dr \right)^2 \leq \int b(r)^2 \gamma(r) r^{d+1} dr$$

that follows from the positivity and normalization of γ via the Schwartz inequality. Therefore the quantity

$$y(t) = \int_0^\infty b(r,t)\gamma(r,t) r^{d+1} dr$$

obeys the inequality

$$\frac{d}{dt} y \geq \frac{d-2}{2d} y^2.$$

Consequently this quantity becomes infinity not later than $t = T_\infty$ where

$$T_\infty = \frac{2d}{d-2} \frac{1}{y(0)}.$$

Now we will choose T and the initial data so that

$$T_\infty < T$$

and conclude that we have blowup in $d > 2$. Recalling that

$$y(0) = d c_{d+2} \int_0^\infty T^{-\frac{d+2}{2}} \exp\left(-\frac{r^2}{4T}\right) m_0(r) r dr$$

using the definition of m and changing the order of integration we obtain that

$$y(0) = 2d c_{d+2} T^{-\frac{d}{2}} \int_0^\infty \rho_0(r) \exp\left(-\frac{r^2}{4T}\right) r^{d-1} dr.$$

By choosing first $T > 0$ and then ρ_0 appropriately, for instance

$$\rho_0(r) = N$$

for $r \leq T^{-\frac{1}{2}}$ and $\rho_0 = 0$ for the rest we deduce that $T_\infty < T$ follows from

$$N d c_{d+2} \int_0^1 \exp\left(-\frac{x^2}{4}\right) x^{d-1} dx > \frac{1}{T(d-2)}$$

i.e., if NT is sufficiently large. If $d = 2$ the analysis is based on the slightly different inequality

$$\left(\int_0^\infty b(r,t)\gamma(r,t) r^3 dr \right)^2 \leq C \int_0^\infty b(r,t)^2 \gamma(r,t) \frac{r^2}{2(T-t)} r^3 dr$$

with $C = 4c_4$ but otherwise it is completely similar.

□

Note that the *blowup quantity* $y(t)$ is actually a priori bounded in terms of the conserved mass! How can this be? Its time derivative is not bounded, so the quantity $y(t)$ develops a discontinuity (shock) before the time T_∞. The above proof is for the blowup of radial solutions. Biler *et. al.* [15] prove blowup without assuming spherical symmetry using a different technique.

In the remainder of this section, we will work with in the coordinate $\chi = r^d/d$. We are interested in solutions on bounded domains Ω with a finite mass so that $\rho \in L^1(\Omega)$. Since $\rho = \partial_\chi m$, we can write Eq. (10) as

$$\partial_t m = -\Gamma = K_d \chi^\alpha \partial_\chi \rho + m\rho, \tag{42}$$

where $\Gamma(\chi)$ is the mass flux through a sphere of radius $(d\chi)^{1/d}$. Differentiating Eq. (42) with respect to χ, we obtain

$$\partial_t \rho = -\partial_\chi \Gamma = \partial_\chi \left(K_d \chi^\alpha \partial_\chi \rho + \rho m \right). \tag{43}$$

4.2. Blowup at the origin

We will show that, if the density ρ does blow up, then it has to blow up at the origin.

Theorem 4. *If there exist $T > 0$, $\epsilon > 0$ and $K < \infty$ such that*

$$\sup_{\chi \le \epsilon, t \in [0,T]} \rho(\chi, t) \le K,$$

then it follows that

$$\sup_{\chi, t \in [0,T]} \rho(\chi, t) < \infty.$$

This theorem implies that, if there is a blowup of the density as $t \to T$, then there exist sequences t_n and χ_n such that $t_n \to T$, $\chi_n \to 0$ and $\limsup_n \rho(\chi_n, t_n) = \infty$.

Proof. We will henceforth denote the derivatives $\partial_\chi f(\chi)$ by f_χ. Multiplying Eq. (43) by ρ^{p-1} and integrating the right hand side of the equation by parts, we obtain:

$$\frac{1}{p}\frac{d}{dt}\int \rho^p \, d\chi = -\frac{4K_d(p-1)}{p^2}\int \chi^\alpha (\rho^{p/2})_\chi^2 \, d\chi - (p-1)\int \rho^{p-1} m \rho_\chi \, d\chi \tag{44}$$

The boundary terms are zero because of the no mass flux boundary conditions. We will now estimate the second term on the right hand side of Eq. (44). We have

$$\begin{aligned} \left| (p-1)\int \rho^{p-1} m \rho_\chi \, d\chi \right| &= \left| \frac{2(p-1)}{p} \int (\rho^{p/2})_\chi \rho^{p/2} m \, d\chi \right| \\ &\le \frac{2K_d(p-1)}{p^2}\int \chi^\alpha (\rho^{p/2})_\chi^2 \, d\chi + C\int \rho^p m^2 \chi^{-\alpha} \, d\chi \end{aligned} \tag{45}$$

where C is a constant. By the hypothesis of the theorem,

$$\sup_{\chi \leq \epsilon} \rho(\chi) = K < \infty.$$

Therefore, using the fact that $m(\chi) \leq M$, we have

$$m^2 \chi^{-\alpha} \leq \max(K\epsilon^{2-\alpha}, M\epsilon^{-\alpha}) < \infty.$$

Combining the above result with Eqs. (44) and (45), we obtain

$$\frac{1}{p}\frac{d}{dt}\int \rho^p \, d\chi + \frac{k_1}{2}\int (\rho^{p/2})_\chi^2 \, d\chi \leq (k_2 M^2)\int \rho^p \, d\chi. \tag{46}$$

where k_1 and k_2 are arbitrary constants.

Using Eq. (46) with the fact that

$$\rho \in L^1(\chi),$$

it is a standard argument to show that the norms

$$W_p = \left(\int \rho^p \, d\chi\right)^{1/p},$$

are *a priori* bounded. This implies that, under the hypothesis of the theorem, for any $\chi \neq 0$, the density $\rho(\chi)$ is bounded.

This proves our claim that if there is blowup, the density at the origin should diverge. □

4.3. Regularity off the origin

We will assume that the solution $\rho(\chi, t)$ blows up at $t = T < \infty$. A point χ is a *blowup point* if there exist sequences $\chi_n \to \chi$ and $t_n \to T$ such that $\limsup \rho(\chi_n, t_n) = \infty$. By our previous result, if $\chi \neq 0$, there should also exist a sequences $\chi_n' \to 0$ and $t_n' \to T$ such that $\limsup \rho(\chi_n', t_n') = \infty$.

It is clear from Eq. (43) that the number of maxima of $\rho(\chi, t)$ is non-increasing in time. We will henceforth assume that the initial condition $\rho_0(\chi)$ has N (a finite number of) local maxima $k_1, k_2, \ldots, k_N$. This will be the case, for example, for smooth initial conditions on bounded domains. For any $K < \infty, t < T$, the set $s(K,t) = \{\chi : \rho(\chi,t) \geq K\}$ is a union of at most N disjoint closed intervals. Since $\rho(\chi,t) \in L^1(\chi) \quad \forall t \leq T$, the Chebyshev inequality implies that $|s(K,t)| \leq M/K$, where $|s|$ is the Lebesgue measure of the set s and $M = \int \rho_0 \, d\chi$. Further, it is true that each of the disjoint intervals in $s(K,t)$ contains at least one local maximum.

Using $|s(K,t)| \to 0$ as $K \to \infty$, noting that the number of local maxima is non-increasing and it is therefore possible to follow a given local maximum in time, and by choosing an appropriate subsequence, we have the following observation:

Lemma 4.1. *If χ is a blowup point there exists a sequence $t_n \to T$ and a local maximum k_i such that $\lim_{n\to\infty} k_i(t_n) = \chi$.*

This Lemma enables us to completely characterize the set of blowup points in terms of the local maxima of $\rho(\chi, t)$.

Theorem 5. *We will assume that:*

(i) $\rho_0(\chi) \in L^1(\chi)$ *and the number of local maxima of* $\rho_0(\chi)$ *is* $N < \infty$.

(ii) The solution $\rho(\chi, t)$ *blows up at* $t = T$.

(iii) For every local maximum k_i *with* $\limsup_{t\to T} \rho(k_i(t), t) = \infty$, *the limit*

$$\lim_{t\to T} k_i(t)$$

exists.

Then, the density can only blowup at the origin ($\chi = 0$).

This theorem implies that if the local maxima of $\rho(x, t)$ do not oscillate very rapidly, in particular, if the velocity of a local maximum dk_i/dt is bounded away from $+\infty$, the solution can only blow up at the origin.

Proof. Assume that we are given a real number $c > 0$. Let K_1 be the maximum value of the density over $0 \le t \le T$ at all the local maxima where the density stays bounded in this interval. Let $K_2 = 6MN/c$ and let $t_0 < T$ be such that for all the local maxima where the density is not bounded in $0 \le t \le T$, we have $|k_i(t) - \lim_{t\to T} k_i(t)| \le c/6N$ for all $t_0 \le t \le T$. Choose $K = \max(K_1, K_2)$. Then, we have $|s(K, t)| \le c/6N$ and

$$\left| \bigcup_{t\in[t_0,T]} s(K, t) \right| \le 2c/3.$$

Consequently, there exist real numbers $0 < a < b < c$ such that $\rho(\chi, t) \le K \quad \forall \chi \in [a, b], \forall t \in [t_0, T]$.

We will choose a function $\phi \in C^\infty(\chi)$ with the following properties

(i) $\phi(\chi) = 0$ for $\chi \le a$.

(ii) $\phi(\chi) = 1$ for $\chi \ge b$.

(iii) $|\phi'(\chi)| \le C\sqrt{\phi(\chi)}$ for all χ.

We will henceforth denote the derivatives $\partial_\chi f(\chi)$ by f_χ. Multiplying Eq. (43) by $\rho^{p-1}\phi$ and integrating the right hand side of the equation by parts, we obtain:

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int \rho^p \phi \, d\chi = & -\frac{4K_d(p-1)}{p^2}\int \chi^\alpha (\rho^{p/2})_\chi^2 \phi \, d\chi - (p-1)\int \rho^{p-1} m \rho_\chi \phi \, d\chi \\ & - K_d \int \chi^\alpha \rho^{p-1} \rho_\chi \phi_\chi \, d\chi - \int \rho^p m \phi_\chi \, d\chi \end{aligned} \tag{47}$$

The boundary terms are zero because of the no mass flux boundary conditions. We will now estimate each term on the right hand side of Eq. (47). The first term gives

$$\begin{aligned}\frac{4K_d(p-1)}{p^2}\int \chi^\alpha(\rho^{p/2})_\chi^2\phi\,d\chi &\geq \frac{4K_d(p-1)}{p^2}\int_{\chi\geq a}\chi^\alpha(\rho^{p/2})_\chi^2\phi\,d\chi\\ &\geq \frac{4K_d(p-1)a^\alpha}{p^2}\int(\rho^{p/2})_\chi^2\phi\,d\chi \end{aligned}\tag{48}$$

The second term can be estimated as

$$\begin{aligned}\left|(p-1)\int\rho^{p-1}m\rho_\chi\phi\,d\chi\right| &= \left|\frac{2(p-1)}{p}\int(\rho^{p/2})_\chi\rho^{p/2}m\phi\,d\chi\right|\\ &\leq \frac{K_d(p-1)a^\alpha}{p^2}\int(\rho^{p/2})_\chi^2\phi\,d\chi + C_1\int\rho^p m^2\phi\,d\chi\\ &\leq \frac{K_d(p-1)a^\alpha}{p^2}\int(\rho^{p/2})_\chi^2\phi\,d\chi + C_1M^2\int\rho^p\phi\,d\chi,\end{aligned}\tag{49}$$

where C_1 is a constant and M is the total mass of the solution. The third term can be bounded as follows

$$\begin{aligned}\left|K_d\int\chi^\alpha\rho^{p-1}\rho_\chi\phi_\chi\,d\chi\right| &= \frac{2K_d}{p}\left|\int_a^b\chi^\alpha\rho^{p/2}(\rho^{p/2})_\chi\phi_\chi\,d\chi\right|\\ &\leq \frac{2K_db^\alpha}{p}\left|\int_a^b\rho^{p/2}(\rho^{p/2})_\chi\phi_\chi\,d\chi\right|\\ &\leq \frac{K_d(p-1)a^\alpha}{p^2C}\int(\rho^{p/2})_\chi^2\phi_\chi^2\,d\chi + \frac{C_2}{b-a}\int_a^b\rho^p\,d\chi,\\ &\leq \frac{K_d(p-1)a^\alpha}{p^2}\int(\rho^{p/2})_\chi^2\phi\,d\chi + C_2K^p,\end{aligned}\tag{50}$$

where C_2 is a constant that depends on a, b, C, K_d and p. We have used item (iii) from the list of properties of ϕ and the fact that $\phi' = 0$ for $\chi \notin [a, b]$ in line 1 in the estimate in Eq. (50). We estimate the last term in Eq. (47) by

$$\begin{aligned}\left|\int\rho^p m\phi_\chi\,d\chi\right| &\leq C\int_a^b\rho^p m\,d\chi\\ &\leq MC(b-a)K^p = C_3K^p,\end{aligned}\tag{51}$$

where we have used $\phi' \leq C\sqrt{\phi}, \phi \leq 1$ and $m \leq M$. Combining the results in Eqs. (48), (49), (50) and (51), we can rewrite Eq. (47) as

$$\frac{1}{p}\frac{d}{dt}\int\rho^p\phi\,d\chi + k_1\int(\rho^{p/2})_\chi^2\phi\,d\chi \leq k_2\int\rho^p\phi\,d\chi + k_3K^p,\tag{52}$$

where

$$k_1 = \frac{2K_d(p-1)a^\alpha}{p^2},$$

$$k_2 = C_1M^2$$

and

$$k_3 = C_2 + C_3.$$

Direct computation yields

$$(\rho^{p/2}\sqrt{\phi})_\chi = (\rho^{p/2})_\chi\sqrt{\phi} + \rho^{p/2}(\sqrt{\phi})_\chi.$$

From the definition of ϕ and using $\phi \in C^\infty(\chi)$, it is true that $(\sqrt{\phi})_\chi = 0 \quad \forall\chi \notin [a,b]$ and $(\sqrt{\phi})_\chi \leq C/2 \quad \forall\chi \in [a,b]$. Therefore, we have

$$\begin{aligned}\int (\rho^{p/2}\sqrt{\phi})_\chi^2\, d\chi &\leq 4\int (\rho^{p/2})_\chi^2\phi\, d\chi + 4\int_a^b \rho^p((\sqrt{\phi})_\chi)^2\, d\chi \\ &\leq 4\int (\rho^{p/2})_\chi^2\phi\, d\chi + C^2(b-a)K^p. \end{aligned} \tag{53}$$

We use Eq. (53) to rewrite Eq. (52) as

$$\frac{1}{p}\frac{d}{dt}\int \rho^p\phi\, d\chi + \frac{k_1}{4}\int (\rho^{p/2}\sqrt{\phi})_\chi^2\, d\chi \leq k_2\int \rho^p\phi\, d\chi + k_4K^p, \tag{54}$$

where $k_4 = k_3 + C^2(b-a)/4$. Using Eq. (54) along with the fact that $\rho(\chi,t) \in L^1(\chi)$, it is a standard argument to show that the norms

$$W_p = \left(\int \rho^p\phi\, d\chi\right)^{1/p},$$

are *a priori* bounded. In particular, this rules out the possibility of any point $\chi > c$ being a blowup point.

The real number $c > 0$ was chosen arbitrarily. This implies that, if the hypothesis of the theorem are satisfied, the density $\rho(\chi)$ for any $\chi \neq 0$ is bounded, and there is no blowup off the origin. □

Remark 1 From the proof, it is clear that if $\chi = c \neq 0$ is a blowup point, the set of blowup points should be dense in $[0,c]$. Since the set of blowup points is closed, as can be easily verified, this implies that the closed interval $[0,c]$ will be contained in the set of blowup points, *i.e.* if $\chi = c \neq 0$ is a blowup point, then every point $0 \leq \chi \leq c$ is also a blowup point.

Remark 2 In Ref. [14], Nagai proves that the solutions are regular off the origin for a system of equations that is very similar to (1) but with a different set of boundary conditions.

4.4. Regularity of the solution for $d < 2$

In section 3.1, we argued that for $d < 2$, given any initial density distribution $\rho(r,t_0)$ in $L^1(\Omega) \cap L^\infty(\Omega)$ that is spherically symmetric, we can find a steady solution $m_S(r)$ such that

$$\int_0^r \rho(r,t_0)r^{d-1}\, dr = m(r,t_0) \leq m_S(r),$$

for all r. The comparison principle implies that $m(r,t) \leq m_S(r)$ for all r and t. In particular, this would imply that

$$\rho(0,t) = \lim_{r\to 0} \frac{dm(r,t)}{r^d} \leq \lim_{r\to 0} \frac{dm_S(r)}{r^d} = \rho_S(0).$$

Combining this with our result that the density is *a priori* bounded off the origin if the density at the origin stays finite, will suggest that there can be no blowup for $d < 2$. We will now prove that this is the case and that the solutions $\rho(r,t)$ for $d < 2$ are in $L^\infty[(0,\infty); L^\infty(\Omega)]$.

Multiplying Eq. (43) by ρ^{p-1} and integrating the right hand side of the equation by parts, we obtain:

$$\frac{1}{p}\frac{d}{dt}\int \rho^p \, d\chi = -\frac{4K_d(p-1)}{p^2}\int \chi^\alpha (\rho^{p/2})_\chi^2 \, d\chi - (p-1)\int \rho^{p-1} m \rho_\chi \, d\chi \tag{55}$$

The boundary terms are zero because of the no mass flux boundary conditions. We will now estimate the second term on the right hand side of Eq. (55). We have

$$\begin{aligned}\left|(p-1)\int \rho^{p-1} m \rho_\chi \, d\chi\right| &= \left|\frac{2K_d(p-1)}{p}\int (\rho^{p/2})_\chi \rho^{p/2} m \, d\chi\right| \\ &\leq \frac{2(p-1)}{p^2}\int \chi^\alpha (\rho^{p/2})_\chi^2 \, d\chi + C_1 \int \rho^p m^2 \chi^{-\alpha} \, d\chi\end{aligned} \tag{56}$$

where C_1 is a constant. Combining Eqs. (55) and (56), we have

$$\frac{1}{p}\frac{d}{dt}\int \rho^p \, d\chi + k_1 \int \chi^\alpha (\rho^{p/2})_\chi^2 \, d\chi \leq k_2 \int \rho^p m^2 \chi^{-\alpha} \, d\chi, \tag{57}$$

where k_1 and k_2 are positive (nonzero) constants. Using the inequality [16]

$$\int_0^\infty x^{-r} F(x)^p \, dx \leq \left(\frac{p}{r-1}\right)^p \int_0^\infty x^{-r} (xF'(x))^p \, dx,$$

for $p \neq 1, r > 1$, we have

$$\int \chi^{\alpha-2} (\rho^{p/2})^2 \, d\chi \leq k_3 \int \chi^\alpha (\rho^{p/2})_\chi^2 \, d\chi. \tag{58}$$

For $d < 2$, we have

$$\alpha = 2 - 2/d < 1,$$

so that $-\alpha > \alpha - 2$. Therefore, we can estimate the term on the right hand side of Eq. (57) by

$$\begin{aligned}\int \rho^p m^2 \chi^{-\alpha} \, d\chi &\leq \int \rho^p m^2 \left(k_4 + \frac{k_1}{2k_3}\chi^{\alpha-2}\right) d\chi \\ &= k_4 \int \rho^p m^2 \, d\chi + \frac{k_1}{2k_3}\int \rho^p m^2 \chi^{\alpha-2} \, d\chi \\ &\leq k_4 M^2 \int \rho^p d\chi + \frac{k_1}{2}\int \chi^\alpha (\rho^{p/2})_\chi^2 \, d\chi,\end{aligned} \tag{59}$$

where k_4 is a constant. We have used Eq. (58) and the fact that the total mass is bounded. Combining Eq. (59) and (57), we obtain

$$\frac{1}{p}\frac{d}{dt}\int \rho^p \, d\chi + \frac{k_1}{2}\int \chi^\alpha (\rho^{p/2})_\chi^2 \, d\chi \leq (k_4 M^2)\int \rho^p \, d\chi. \tag{60}$$

Eq. (60) with the fact that $\rho \in L^1(\chi)$ implies that the norms

$$V_p = \left(\int \rho^p \, d\chi\right)^{1/p},$$

are all *a priori* bounded demonstrating the regularity of all solutions for $d < 2$.

5. Blowing up solutions

In section 4.1, we show that there exist initial conditions for the density $\rho(r,0)$ in $d \geq 2$ such that the solution has a finite time singularity where the $L^{d/2}$ norm of $\rho(r,t)$ diverges. We will analyze some of these solutions in this section.

5.1. Self-similar blowup

The scale invariance

$$r \to r/L, \rho_L(r,t) = L^{-2}\rho(r/L, t/L^2)$$

motivates us to look for solutions with the natural scaling

$$L \sim \sqrt{T-t}. \tag{61}$$

for solutions that blow up at $t = T$. We look for solutions of the form

$$\rho(r,t) \sim \frac{1}{T-t}\Pi\left(\frac{r}{\sqrt{T-t}}\right) \tag{62}$$

subject to the boundary conditions

$$\Pi(\eta) \to 0 \text{ as } \eta \to \infty \quad \text{and} \quad \Pi(0) < \infty. \tag{63}$$

To look for solutions that blow up, we rewrite the equations in terms of the scaled radial coordinate

$$\eta = r/\sqrt{T-t}$$

and the slow time

$$\tau = -\log(T-t),$$

and we obtain

$$\partial_\eta^2\Pi + \frac{d-1}{\eta}\partial_\eta\Pi + \frac{1}{\eta^{d-1}}\partial_\eta(\tilde{m}(\eta)\Pi) - \Pi - \frac{1}{2}\eta\partial_\eta\Pi = 0, \tag{64}$$

where $\tilde{m}(\eta) = \int_0^\eta \xi^{d-1}\Pi(\xi)d\xi$.

It is again convenient to solve for the function $h(r,t)$ instead of $\rho(r,t)$. Keeping in mind the rescaling properties of $h(r,t)$, we write $h(r,t) = H(\eta,\tau)$. We imagine that $H(\eta,\tau) = H_S(\eta) + H_C(\eta,\tau)$, where $H_S(\eta)$ is a self similar solution that describes

the blowup and $H_C(\eta,\tau)$ describes the approach to the self similar solution, so that $H_C(\eta_0,\tau)\to 0$ as $\tau\to\infty$ for any fixed $\eta_0<\infty$. In the rescaled coordinates, we can rewrite Eq. (14) as

$$\partial_\tau H(\eta,\tau)=\partial_\eta^2 H(\eta,\tau)+\left(\frac{d-3+2H(\eta,\tau)}{\eta}\right)\partial_\eta H(\eta,\tau) + \frac{2(d-2)H(\eta,\tau)}{\eta^2}(H(\eta,\tau)-1)-\frac{\eta}{2}\partial_\eta H(\eta,\tau) \tag{65}$$

The τ independent solutions of this equation that satisfy the boundary conditions

$$\eta\partial_\eta H_S(\eta)\to 0 \text{ as } \eta\to\infty \quad \text{and} \quad \lim_{\eta\to 0}\frac{H_S(\eta)}{\eta^2}<\infty,$$

give exactly self-similar solutions that blow up at a finite time T. If we set $\partial_\tau H_S(\eta)=0$ in (65), we obtain

$$\partial_\eta^2 H_S+\left[\frac{d-3+2H_S}{\eta}-\frac{\eta}{2}\right]\partial_\eta H_S+\frac{2(d-2)(H_S-1)H_S}{\eta^2}=0. \tag{66}$$

Besides satisfying the boundary conditions at infinity and at $\eta=0$, in order to describe the blowup of a solution whose initial condition is a non-negative density $\rho_o(r)$, we need that the solution H_S should have a non-negative density and a non-negative mass for all $\eta\geq 0$.

Fig. 9 shows numerically obtained solutions to Eq. (66) that satisfy the boundary conditions and also have a non-negative mass and density.

We have the following theorem about the solutions to Eq. (66).

Theorem 6. *For $2<d<10$, there exist a countable number of solutions to the equation*

$$\partial_\eta^2 H_S+\frac{d-3+2H_S}{\eta}\partial_\eta H_S+\frac{2(d-2)H_S}{\eta^2}(H_S-1)-\frac{\eta}{2}\partial_\eta H_S=0.$$

satisfying the boundary conditions $H_S(\eta)/\eta^2$ is bounded as $\eta\to 0$ and $\eta\partial_\eta H_S\to 0$ as $\eta\to\infty$. These solutions also have the properties that $H_S(\eta)\geq 0$ and $G(\eta)=\partial_\eta(\eta^{d-2}H_S(\eta))\geq 0$ for all η.

These solutions are characterized by the number of intersections with the line $H=1$. More precisely, given a non-negative integer N, there exists at least one $U\in[-1,1]$ and a solution H_N such that the solution $H_N(\eta)$ intersects the line $H=1$ precisely N times and $\lim_{\eta\to\infty}H(\eta)=1+U$.

We will prove this theorem in the Appendix.

Remark 1 The theorem as stated guarantees the existence of a solution H_N having N intersections with the line $H=1$ for any given integer $N\geq 0$, but it does not make any claims on the uniqueness of such solutions. Numerical evidence suggests that the solution H_N with $U=\lim_{\eta\to\infty}H_N(\eta)-1\in[-1,1]$ is unique. If, however, we relax the restriction that $U\in[-1,1]$, it turns out that the solutions are not necessarily unique. We will return to this point subsequently.

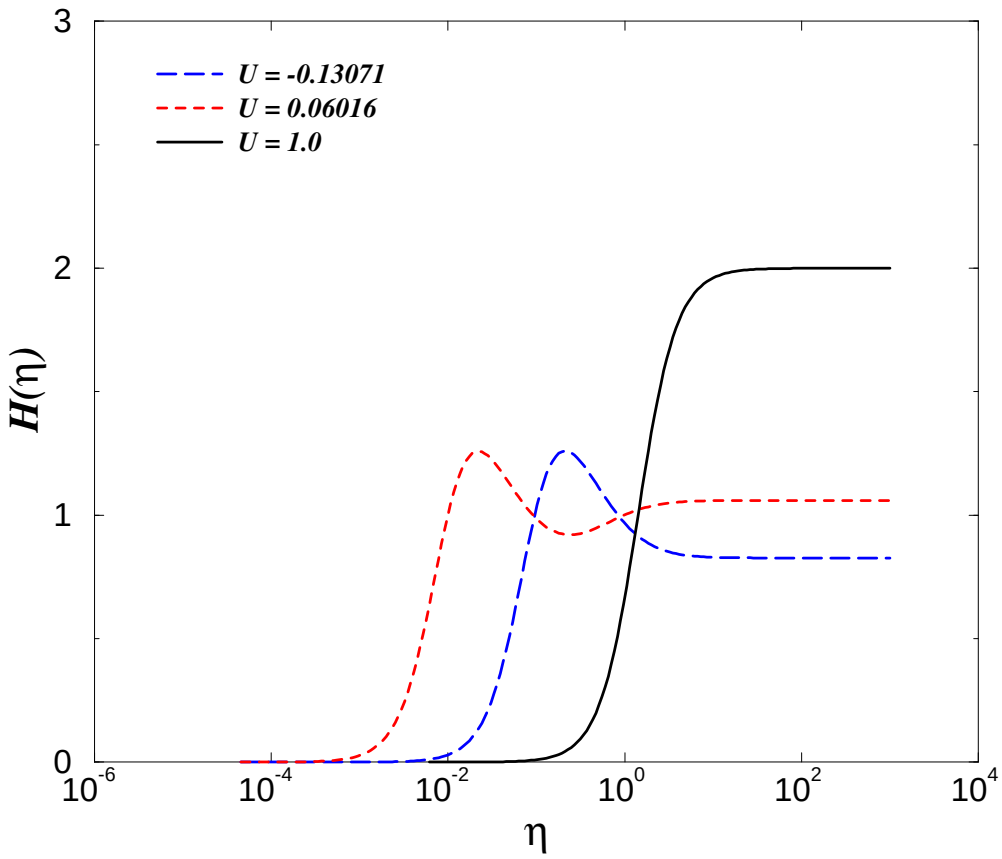

Figure 9. Similarity solutions for the blowup profiles for $d = 3$. The similarity solutions satisfy $\lim_{\eta\to\infty} H(\eta) = 1 + U$. The figure shows solutions with $N = 1, 2$ and 3 intersections with the line $H = 1$.

Remark 2 It is easily seen that, for $2 < d$, all the solutions to Eq. (66) satisfying the boundary conditions describe blowup where the density diverges at the origin but no mass is concentrated at the origin, *i.e.*,

$$\lim_{\epsilon\to 0}\left(\lim_{t\to T} m(\epsilon, t)\right) = 0.$$

For all $d > 2$, we can check that

$$H_1(\eta) = \frac{2\eta^2}{2(d-2) + \eta^2} \tag{67}$$

is a solution of Eq. (66) satisfying the boundary conditions and the non-negativity requirements. This solution has $U = 1$ and $N = 1$ (one intersection with the line $H = 1$.)

We will show below that all the stable self-similar blowup profiles should have no extrema for $\eta \in (0, \infty)$ so that they can have no more than one intersection with the

line $H = 1$. Therefore, all the other solutions to Eq. (66) for $2 < d < 10$, that are described in Theorem 6 give blowup profiles that are unstable.

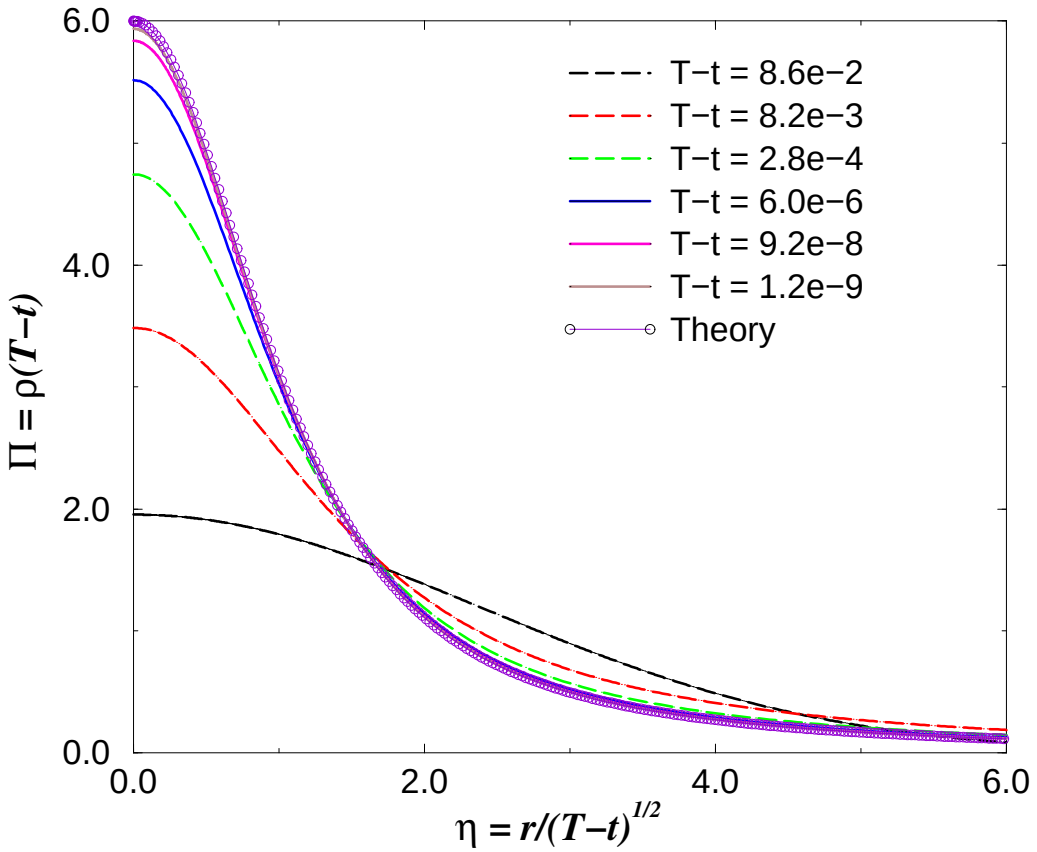

Figure 10. Blowup profile for generic initial data for $d = 3$. The solution converges in similarity variables to the theoretical profile given by Π_1. The blowup time T is estimated using the density at the origin.

Fig. 10 shows numerically obtained solutions to Eq. (1) with spherical symmetry for $d = 3$. The figure shows the density scaled by the inverse of the time to blowup as a function of the radial coordinate rescaled by the square root of the time to blowup. This is precisely the scaling in Eq. (62). The theoretical curve is given by the density corresponding to the solution $H_1(\eta)$ in Eq. (67), so that

$$\Pi_1(\eta) = \frac{4(d-2)(2d+\eta^2)}{[2(d-2)+\eta^2]^2}. \tag{68}$$

We find that this blowup profile is generic in the sense that there is an open set of initial conditions all of which blow up converging to this profile.

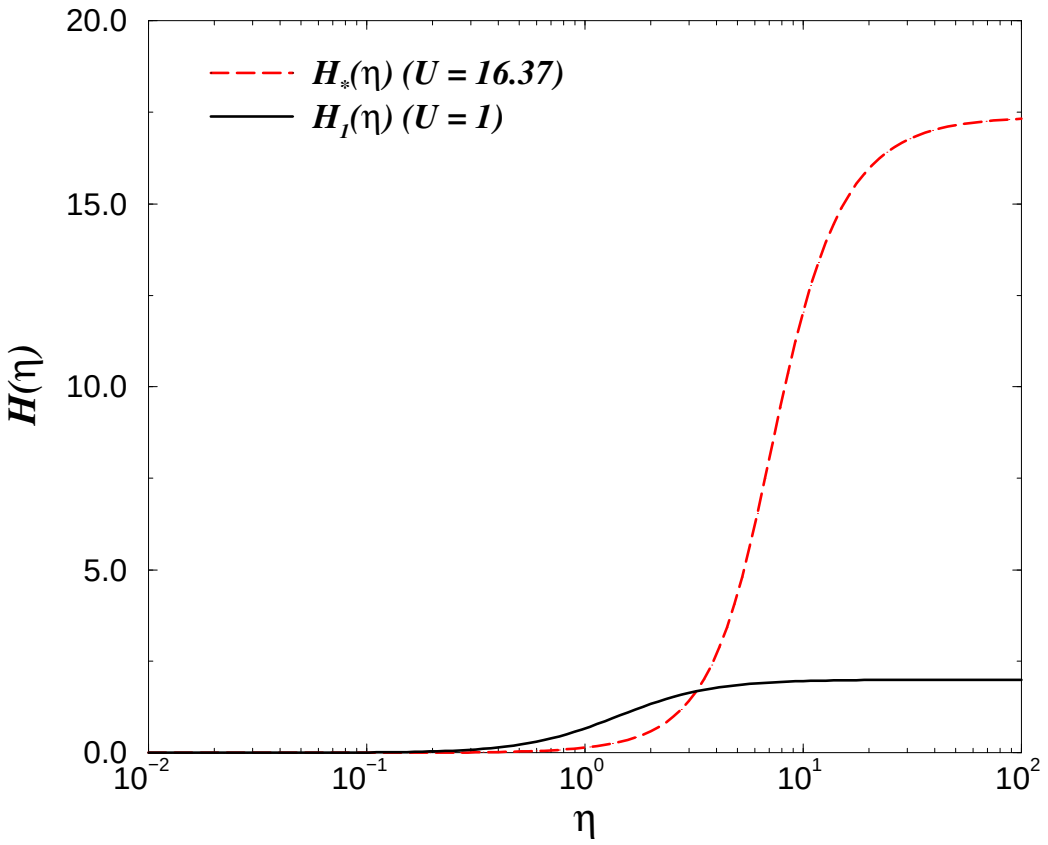

Figure 11. Similarity solutions for $d = 3$. This figure shows the solutions $H_1(\eta)$ and $H_*(\eta)$ that have $N = 1$.

As we remarked earlier, the solutions with $N = 1$ are not unique if we relax the restriction that $U \in [-1, 1]$. Fig. 11 shows $H_*(\eta)$, a solution of Eq. (66) for $d = 3$ satisfying the boundary conditions with $U = 16.378\ldots$. Also shown is the solution $H_1(\eta)$ given by (67). Both these solutions have $N = 1$ and have no extrema for $\eta \in (0, \infty)$ in contrast to the solutions $H_N(\eta)$ with $N \geq 2$ whose existence is guaranteed by Theorem 6.

5.2. Non Self-similar blowup

In this section, we present numerical results about certain solutions $\rho(r, t)$ that blow up in a non self-similar fashion. By non self-similar, we mean that the asymptotics of such solutions do not develop on scales corresponding to the similarity variables $\eta = r/\sqrt{T - t}$ and $\Pi = (T - t)\rho$. They correspond to an imploding smoothed out shock wave that collapses into a Dirac mass at the origin when the singularity is formed. Hence, unlike the self-similar solutions described above, a finite portion of the initial mass is driven

to the origin at the blowup time. For $d = 3$, those solutions have been analyzed in [11] by means of matched asymptotics.

As mentioned in the previous section, the density profile Π_1 given by the explicit similarity solution in (68) has been obtained numerically for several initial data with the density concentrated in balls around the origin or in shells close enough to the origin. On the other hand, starting with an initial condition whose mass is concentrated in a shell away from the origin, the numerical integration of Eq. (8) does not lead to this self-similar blowup, but to the formation of a peak moving towards the origin, with a height blowing up to infinity and a width shrinking to zero at a finite time.

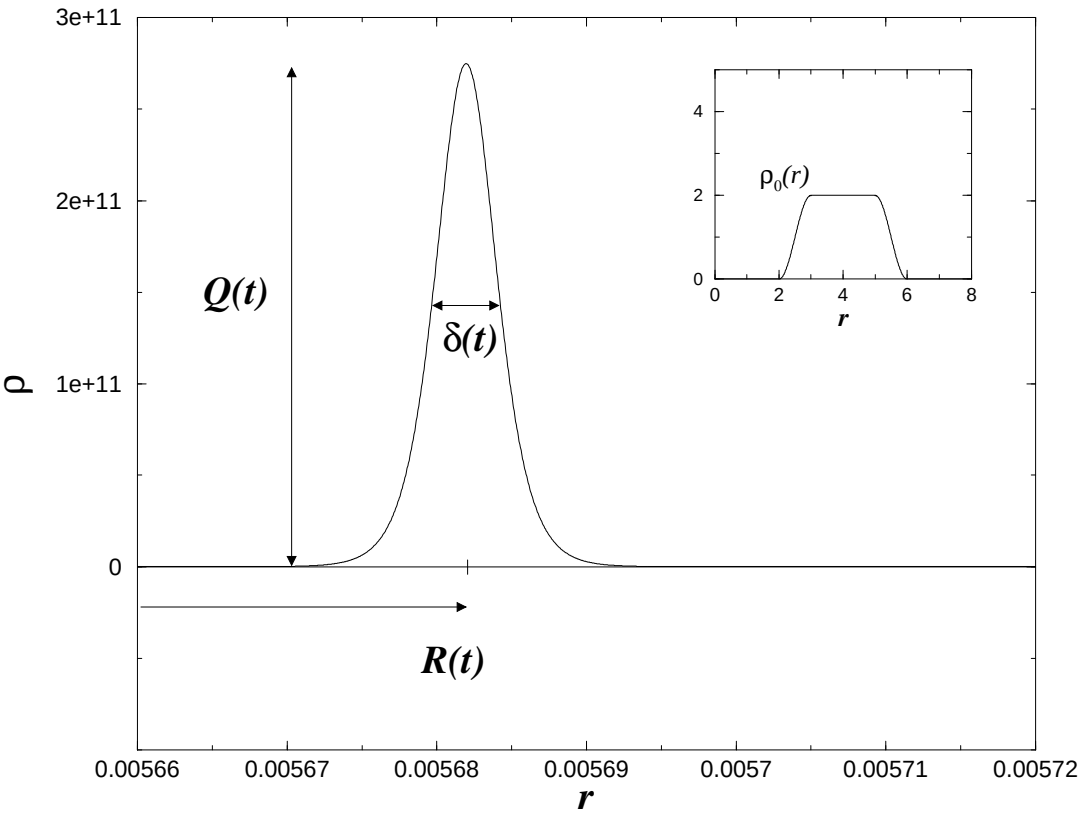

Figure 12. The density profile in $d = 3$ at time to blowup $T - t = 7.13 \times 10^{-9}$ for the initial condition shown in the upper right corner.

This behavior is also generic in the sense that it has been observed for various such initial data, so that it is expected for an open set of initial conditions. Fig. 12 shows a typical profile just prior blowup. Furthermore, the dynamics of the height $h(t)$ of the peak, its width $\delta(t)$ and its location $R(t)$ depend upon the dimension d and satisfy for

several values of d, $d > 2$, as $t \to T$,

$$h(t) = c_1(T-t)^{-2\frac{d-1}{d}}, \quad \delta(t) = c_2(T-t)^{\frac{d-1}{d}}, \quad R(t) = c_3(T-t)^{1/d}. \quad (69)$$

See Figs. 13, 14 and 15.

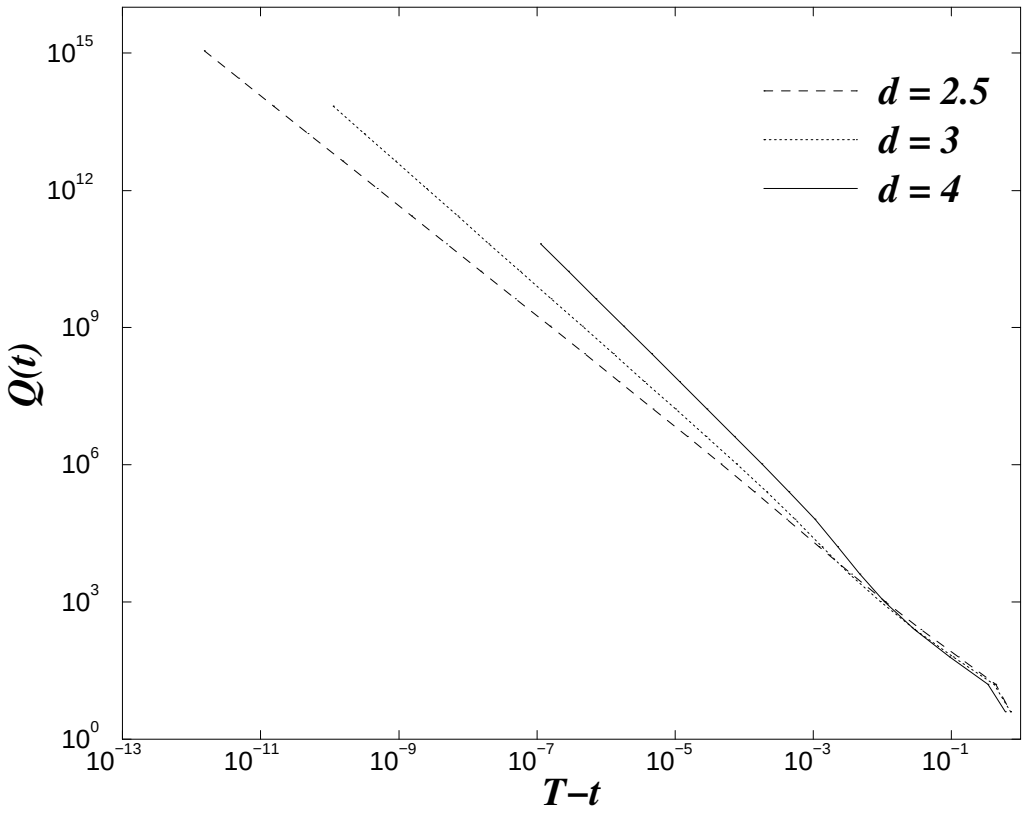

Figure 13. The height $h(t)$ for $d = 2.5, 3$ and 4. In the asymptotic regime, the best fits have slopes of value 1.203, 1.335 and 1.498 respectively, in good agreement with the theoretically predicted values of 6/5, 5/3 and 3/2.

A simple motivation for the power laws in Eq. (69) goes as follows. We begin by assuming that $\delta \ll R \ll 1$ and that the peak contains a finite and slowly changing portion of mass, namely that $m(R,t) \sim R^{d-1}h\delta \sim M$. In the neighborhood of the peak, we have $\partial_r \rho \sim h/\delta$ and $\partial_r^2 \rho \sim h/\delta^2$, so that the second term in the RHS of Eq. (8) becomes asymptotically irrelevant, leading to

$$\partial_t \rho = \partial_r^2 \rho + \frac{1}{r^{d-1}} \partial_r (m\rho). \quad (70)$$

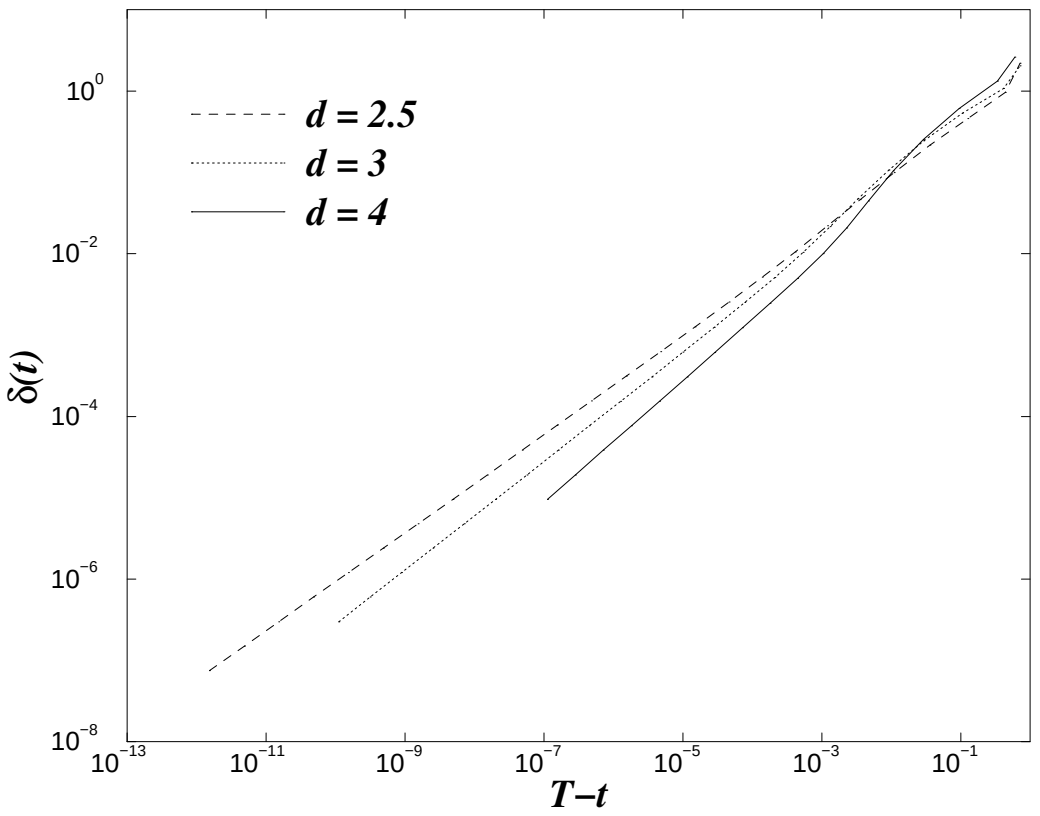

Figure 14. The width $\delta(t)$ for $d = 2.5, 3$ and 4. In the asymptotic regime, the best fits have slopes of value 0.602, 0.668 and 0.749 respectively, in good agreement with the theoretically predicted values of 3/5, 2/3 and 3/4. Here, δ is defined as the width at half the height of the peak.

We will require that the three remaining terms to be of the same size. Recalling that $\partial_r m = r^{d-1}\rho$, we obtain

$$h\delta^2 \sim 1. \tag{71}$$

With the mass conservation assumption, this leads to

$$\delta \sim R^{d-1}. \tag{72}$$

Those relations together with $R \ll 1$ imply for $d > 2$ that (70) is in turn asymptotically equivalent to

$$\partial_t \rho = \partial_r \Big(\partial_r \rho + \frac{m}{r^{d-1}} \rho \Big). \tag{73}$$

To estimate $R(t)$, we next consider traveling wave solutions of Eq. (73) of the form $\rho(r,t) = \varphi(\xi)$, $\xi = r - R(t)$. Assuming that φ and φ' decrease to zero as $\xi \to \infty$ and

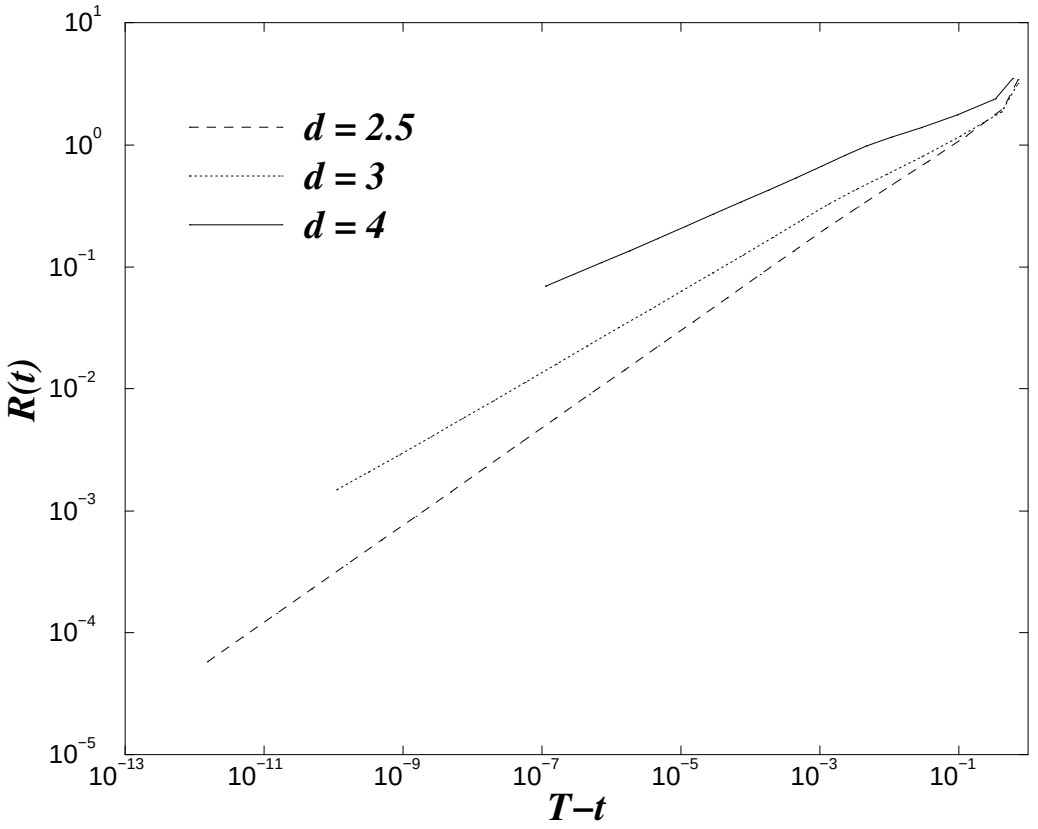

Figure 15. The location $R(t)$ for $d = 2.5, 3$ and 4. In the asymptotic regime, the best fits have slopes of value 0.399, 0.327 and 0.246 respectively, in good agreement with the theoretically predicted values of 2/5, 1/3 and 1/4.

integrating the equation for φ once yields

$$-\dot{R}\varphi = \varphi' + \frac{m}{(\xi + R)^{d-1}}\varphi.$$

At $\xi = 0$, we have $\varphi' = 0$ so that

$$\dot{R} \sim -\frac{M}{R^{d-1}}.$$

This yields the power law for $R(t)$ in Eq. (69). The behaviors of δ and h follow from Eq. (71) and Eq. (72).

5.3. Asymptotic solutions neglecting diffusion

In the previous section, we discussed solutions that lead to blowup on scales different from the parabolic scaling $\eta = r/\sqrt{T-t}$. In this section, we will discuss solutions that

blow up on scales $L(t) \gg \sqrt{T-t}$ with $L(t) \to 0$ as $t \to T$, so that the density blows up at $t = T$ but the diffusion term becomes asymptotically irrelevant. Keeping in mind that the nonlinearity gives $\partial_t \rho \sim \rho^2$, we consider the following *ansatz* for solutions that blow up at $t = T$:

$$\rho(r,t) = \frac{1}{T-t}\Pi(\eta,\tau)$$

where η is the similarity variable $r/L(t)$, $L(t)$ is an as yet unspecified function of t and $\tau = -\log(T-t)$. To describe a blowup we will require that Π satisfy the boundary conditions

$$\Pi(0) < \infty, \quad \Pi(\eta) \to 0 \text{ as } \eta \to \infty$$

with $\Pi(\eta) \geq 0$ for all $\eta \geq 0$. As before, we shall eventually go to a similarity solution by requiring that the solution be τ independent. Using the above *ansatz*, Eq. (11) gives

$$b(r,t) = \frac{1}{T-t}B(\eta,\tau)$$

and Eq. (13) can be written in the form:

$$\partial_\tau B + B - (T-t)\frac{d\log L}{dt}\eta\partial_\eta B = \frac{T-t}{L^2(t)}\bar{\Delta}_{d+2}B + \frac{B}{d}\eta\partial_\eta B + B^2, \tag{74}$$

where $\bar{\Delta}_{d+2}$ is the $d+2$ dimensional Laplacian in η. Using $L(t) \gg \sqrt{T-t}$, we see that the diffusive term is asymptotically irrelevant in comparison to the nonlinearity. Assuming that we approach a τ independent solution for B as $\tau \to \infty$, we obtain the asymptotic equation

$$\left[\frac{B}{d} - x\right]\eta\partial_\eta B = B - B^2, \tag{75}$$

with

$$x = -\lim_{t\to T}(T-t)\frac{d\log L}{dt}.$$

In particular, for $L(t) = (T-t)^y$, we have $x = y$ and $L(t) \gg \sqrt{T-t}$ requires that $x \geq 1/2$ (The case $x = 1/2$ occurs for example with $L(t) = \sqrt{(T-t)|\log(T-t)|}$ which is the scaling for a family of solutions for the semilinear heat equation [6]).

From the boundary conditions specified above, we see that the solution should satisfy

$$B(0) < \infty \quad \text{and} \quad B(\eta) \to 0 \text{ as } \eta \to \infty.$$

Defining $\xi = \log\eta$, Eq. (75) can be written as the autonomous system

$$B_\xi = d\frac{B - B^2}{B - dx}. \tag{76}$$

The fixed point at $B = 0$ is stable, the fixed point at $B = 1$ is repelling, and there are no critical points in the interval $(0,1)$ if $x > 1/d$. Consequently, for $1/d < x \leq 1/2$, given any $C \in (0,1)$, we have a solution of Eq. (76) with $B(\xi = 0) = C$,

$$\lim_{\xi\to-\infty} B = 1 \quad \text{and} \quad \lim_{\xi\to\infty} B = 0,$$

so that it satisfies the boundary conditions. We therefore have a continuous family of solutions for each x that satisfy the boundary conditions. We can integrate Eq. (76) to obtain the implicit solutions

$$\left(\frac{1-B}{B}\right)^{x}(1-B)^{-1/d} = ke^{\xi} = k\eta, \tag{77}$$

where $k \in (0,\infty)$ is an arbitrary constant that parameterizes the continuous family of solutions. These solutions satisfy $B \approx 1 - k_1\eta^{d/(dx-1)}$ for small η and $B \approx k_2\eta^{-1/x}$ for large η where k_1 and k_2 are constants that depend on the parameter k.

We have exhibited continuous families of solutions for each x in $(1/d, 1/2]$ for the asymptotic equation Eq. (75), but we have not observed any of these solutions in our numerical simulations and their role in the dynamics of the system is unclear.

6. Stability and convergence properties of blowup profiles

In Sec. 5.1, we saw that the system has a countable number of possible self-similar blowup profiles with the parabolic scaling. This brings up the question of which of these profiles are stable and which of them would we expect to see if we start from generic initial data.

6.1. Stability of the Blowup profiles

We will now prove the following result concerning the stability of the various blowup profiles.

Theorem 7. *: Every self-similar solution $H_S(\eta)$ that blows up at a finite time has an unstable mode corresponding to changing the blowup time. Also, a blowup profile with n extrema in $(0,\infty)$ has at least n additional unstable modes.*

Proof. Linearizing Eq. (65) about $H = H_S$, we obtain

$$\begin{aligned}\partial_\tau\Phi(\eta,\tau) = \partial_\eta^2\Phi(\eta,\tau) + \left(\frac{d-3+2H_S(\eta)}{\eta} - \frac{\eta}{2}\right)\partial_\eta\Phi(\eta,\tau) \\ + \frac{2\eta\partial_\eta H_S(\eta) + 2(d-2)(2H_S(\eta)-1)}{\eta^2}\Phi(\eta,\tau).\end{aligned} \tag{78}$$

We will consider the eigenvalue problem for the above equation. For a given λ, let Φ_λ be a solution of

$$\begin{aligned}\partial_\eta^2\Phi_\lambda(\eta) + \left(\frac{d-3+2H_S(\eta,t)}{\eta} - \frac{\eta}{2}\right)\partial_\eta\Phi_\lambda(\eta) \\ + \left[\frac{2\eta\partial_\eta H_S(\eta) + 2(d-2)(2H_S(\eta)-1)}{\eta^2} + \lambda\right]\Phi_\lambda(\eta) = 0.\end{aligned} \tag{79}$$

Then, $\Phi_\lambda(\eta)e^{-\lambda\tau}$ is a solution of Eq. (78). Near $\eta = 0$, the leading behaviors of the two linearly independent solutions of (79) are $\Phi_\lambda(\eta) \sim \eta^2$ and $\Phi_\lambda(\eta) \sim 1/\eta^{d-2}$. To have H be nonsingular near $\eta = 0$, we will require the boundary condition

$$\Phi_\lambda(\eta) \to \eta^2 \text{ as } \eta \to 0.$$

For large η, the leading behaviors of the two independent solutions are $\Phi_\lambda(\eta) \sim \eta^{-(2\lambda+1)}e^{\eta^2/4}$ and $\Phi_\lambda(\eta) \sim \eta^{2\lambda}$. Although the functions $\eta^{2\lambda}$ diverge as $\eta \to \infty$, in terms of r and t, we have

$$\eta^{2\lambda}e^{-\lambda\tau} = r^{2\lambda},$$

is bounded for r bounded as $t \to T$. Consequently, the solutions which behave like $\eta^{2\lambda}$ are admissible but the functions that behave like $\eta^{-(1+2\lambda)}e^{\eta^2/4}$ diverge at finite r as $t \to T$ and are consequently inadmissible. We are interested therefore in finding λ_n so that the solution $\Phi_{\lambda_n}(\eta)$ also satisfies the boundary condition

$$\Phi_{\lambda_n}(\eta)e^{-\eta} \to 0 \text{ as } \eta \to \infty.$$

We have the freedom to choose the blowup time T *i.e.*, $H_S(r/\sqrt{T+\delta-t})$ is a solution to Eq. (14) for all δ. Using $\eta = r/\sqrt{T-t}$ and $\tau = -\log(T-t)$, we see that

$$\bar{H}_S(\eta,\tau,\delta) = H_S(r/\sqrt{T+\delta-t}) = H_S\left(\eta\sqrt{\frac{e^{-\tau}}{e^{-\tau}+\delta}}\right)$$

is a solution of Eq. (65) for all δ. Consequently,

$$\begin{aligned}\partial_\tau \bar{H}_S(\eta,\tau,\delta) = \partial_\eta^2 \bar{H}_S(\eta,\tau,\delta) + \left(\frac{d-3+2\bar{H}_S(\eta,\tau,\delta)}{\eta}\right)\partial_\eta \bar{H}_S(\eta,\tau,\delta) \\ + \frac{2(d-2)\bar{H}_S(\eta,\tau,\delta)}{\eta^2}(\bar{H}_S(\eta,\tau,\delta)-1) - \frac{\eta}{2}\partial_\eta \bar{H}_S(\eta,\tau,\delta).\end{aligned} \tag{80}$$

Differentiating the above equation *w.r.t.* δ and setting $\delta = 0$, we obtain

$$\begin{aligned}\partial_\tau e^\tau F_S^0(\eta) = \partial_\eta^2 e^\tau F_S^0(\eta) + \left(\frac{d-3+2H_S(\eta)}{\eta} - \frac{\eta}{2}\right)\partial_\eta e^\tau F_S^0(\eta) \\ + \left[\frac{2\eta\partial_\eta H_S(\eta) + 2(d-2)(2H_S(\eta)-1)}{\eta^2}\right] e^\tau F_S^0(\eta),\end{aligned} \tag{81}$$

where

$$F_S^0(\eta) = \eta\partial_\eta H_S(\eta).$$

We therefore obtain

$$\begin{aligned}\partial_\eta^2 F_S^0(\eta) + \left(\frac{d-3+2H_S(\eta)}{\eta} - \frac{\eta}{2}\right)\partial_\eta F_S^0(\eta) \\ + \left[\frac{2\eta\partial_\eta H_S(\eta) + 2(d-2)(2H_S(\eta)-1)}{\eta^2} - 1\right] F_S^0(r) = 0,\end{aligned} \tag{82}$$

Comparing Eq. (82) with Eq. (79), we see that $F_S^0(\eta)$ is an eigenfunction with an eigenvalue $\lambda = -1$ since it also satisfies the boundary conditions $F_S^0(\eta) \sim \eta^2$ as $\eta \to 0$ and $F_S^0(\eta)e^{-\eta} \to 0$ as $\eta \to \infty$. If the function $H_S(\eta)$ has k extrema in $(0,\infty)$, the function $\Phi_0(\eta)$ has k zeroes in $(0,\infty)$. Now, by standard results from Sturm-Liouville theory, it follows that there exist $\lambda_{-k}^* < \ldots < \lambda_{-2}^* < \lambda_{-1}^* < 0$ such that $\Phi_{\lambda_{-j}^*}(\eta)e^{-\eta} \to 0$ as $\eta \to \infty$ and $\Phi_{\lambda_{-k}^*}(\eta)$ has exactly $k-j$ zeroes in $(0,R)$.

All the eigenfunctions with eigenvalues less than zero are unstable. This proves our claim that all the self-similar blowup profiles $H_S(\eta)$ have an unstable mode corresponding to changing the blowup time. Further, a blowup profile with k extrema in $(0,\infty)$ has at least k additional unstable modes. □

This theorem implies that a blowup profile that is stable to small perturbations in the initial conditions (except for possibly a change in the blowup time) cannot have any extrema for $\eta \in (0,\infty)$.

6.2. Convergence to the stable blowup profile

If we choose the blowup time T correctly in the definition of τ and η, we can kill the mode corresponding to the change in blowup time. From the linear analysis in the preceding section, it is clear that a solution of (65) is asymptotically given by

$$H(\eta,\tau) = H_S(\eta) + aF_S^1(\eta)e^{-\lambda_1\tau} + O(e^{-\lambda_2\tau}), \tag{83}$$

where λ_1 and λ_2 are respectively the second and the third smallest eigenvalues of the Eq. (78). F_S^1 is the eigenfunction corresponding to λ_1 and a is an constant depending on the initial conditions, generically not equal to zero.

If the blowup profile is stable to small perturbations in the initial conditions, it follows from Eq. (83) that

$$\lambda_1 > 0.$$

Given a similarity solution H_S, we can numerically calculate λ_1 from Eq. (78). For $d = 3$ and $H_S(\eta) = H_1(\eta) = 2\eta^2/(2(d-2)+\eta^2)$, we obtain $\lambda_1 = 0.27\ldots$. We also find that $\lambda_1 > 0$ for $H_1(\eta)$ for all $d > 2$. Therefore, the similarity solution

$$H_1(\eta) = \frac{2\eta^2}{2(d-2)+\eta^2},$$

will be seen generically in solutions that blow up according to the parabolic scaling. Using $h(r,t) = H(\eta,\tau)$ and the definition of h, we get

$$\begin{aligned}\rho(r,t) = {} & \frac{4(d-2)(2d(T-t)+r^2)}{(r^2+2(d-2)(T-t))^2} + a\frac{(T-t)^{\lambda_1}}{T-t}\Pi_1^1\left(\frac{r}{\sqrt{T-t}}\right) \\ & + O((T-t)^{\lambda_2-1}),\end{aligned} \tag{84}$$

where

$$\Pi_1^1(\eta) = \frac{1}{\eta^{d-1}}\partial_\eta(2\eta^{d-2}F_1^1(\eta)).$$

Using the fact $F_1^1(\eta) \to \eta^2$ as $\eta \to 0$, we have

$$\rho(0,t) = \frac{1}{(T-t)}\left[\frac{2d}{d-2} + 2ad(T-t)^{\lambda_1} + o((T-t)^{\lambda_2})\right] \tag{85}$$

For $d = 3$, Eq. (85) yields the prediction

$$\rho(0,t) \approx \frac{6}{T-t}\left[1 + C(T-t)^{0.27}\right].$$

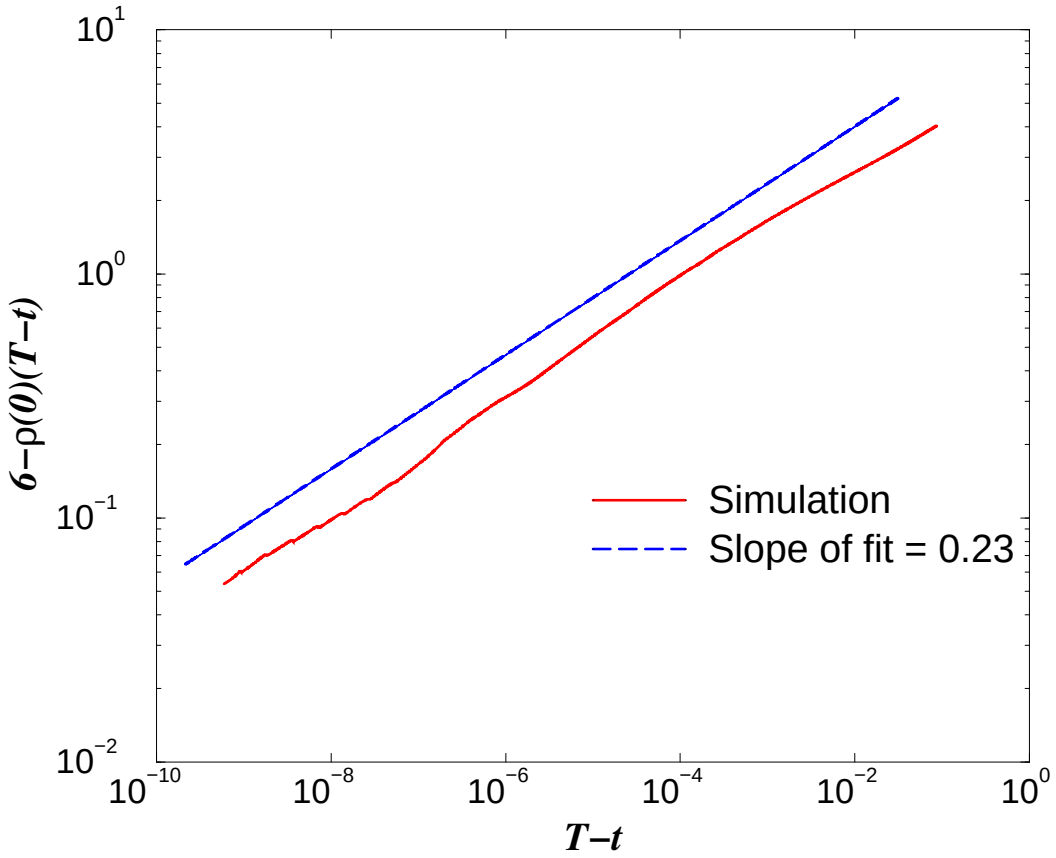

Figure 16. Convergence to the Blowup profile. The slope of the best fit is 0.23 in good agreement with the linear analysis which gives 0.272 . . . . Note that this exponent is small so that the convergence is very slow and one has to solve the equations until very close to the blowup time to see the convergence to the self similar solution.

where the constant C is generically not equal to zero.

Fig. 16 is a plot generated by numerically solving the system in Eq. (1) and it agrees well with the prediction.

In Sec. 5.1, we saw that there exist two solutions to Eq. (66) each having one intersection with the line $H = 1$ and no extrema for $\eta \in (0, \infty)$. In the above discussion we showed that the solution in (67) gives a stable self-similar profile. Numerically calculating λ_1 and λ_2 for the solution $H_*(\eta)$ for $d = 3$, we see that $\lambda_1 = -0.443 \ldots < 0$ and $\lambda_2 > 0$. Therefore, although the solution $H_*(\eta)$ has no extrema for $\eta \in (0, \infty)$, it has one unstable mode (besides the mode that changes the blowup time). This explains why we do not see blowup corresponding to the solution $H_*(\eta)$ in numerical simulations with generic initial data.

6.3. Critical solutions

The analysis in the previous sections has led us to the identification of several generic final states for the solutions of the system in (1) with spherical symmetry. For $d > 2$ and on bounded domains, we see that there exist open sets of initial conditions for which the system can :

(i) Converge to a stable steady state given by $h_0(r)$ as $t \to \infty$ (see Sec. 3.3),

(ii) Blow up at a finite time T converging to the self-similar blowup profile given by $H_1(\eta)$ in Eq. (67) (see Sec. 5.1), or

(iii) Blow up at a finite time T in a non self-similar fashion given by Eq. (69) (see Sec. 5.2).

The fate of an initial condition depends not only on its conserved total mass, but also on the density distribution $\rho_0(r)$. The existence of more than one kind of generic behavior brings up the question of what determines the boundaries (separatrices) between the different sets of initial conditions that have different final states. A related question is what happens to solutions whose initial data lies on the separatrix between the basins of attraction of two different types of stable (generic) behaviors.

We investigate these questions numerically by following the solutions for a family of initial conditions $\rho_0(r, p)$ parameterized by a parameter p. The family of initial conditions is chosen such that the final states for the initial conditions given by $p = p_0$ and $p = p_1$ are different. In that case, there exists at least one critical value $p_c \in (p_0, p_1)$ such that the final states for initial conditions for any p_a and p_b sufficiently close to p_c with $p_a < p_c < p_b$ are different. We find the critical value p_c by successive bisection and we can determine the separatrix by following the solution with initial condition $\rho_0(r, p_c)$ for different families of initial conditions.

We find that the boundaries between initial conditions leading to different kinds of generic behavior are the stable manifolds of solutions with one unstable direction (codimension one). We will refer to these solutions as critical solutions and they are the final states for initial conditions lying on the separatrix between different basins of attraction. If we choose an initial condition $\rho_0(r, p_d)$ with p_d close to the critical value p_c, the solution has a transient phase when it stays close to the stable manifold of the critical solutions and seems to converge to the critical solution. However, it will eventually move away from the critical solution along the unstable direction and end up in one of the generic final states of the system.

For $2 < d < 10$, if the total mass is smaller than the critical mass $M_c(R)$ for a domain of size R (See Eq. (33)), one possible generic final state is the stable steady state given by $h_0(r)$. Numerical evidence suggests that the separatrix $\mathcal{M}_0$ between the steady state and the self similar blowup has two pieces. One piece $\mathcal{M}_a$ is given by the stable manifold of the steady state with one unstable mode $h_1(r)$. Another piece $\mathcal{M}_b$ is given by the set of solutions converging to the self similar blowup profile given by $H_2(\eta)$, whose stable manifold is of codimension one also (if we take out the mode that changes the blowup time).

The existence of two different critical solutions suggests that there should exist a codimension two manifold $\mathcal{M}_1$ that serves to separate the initial conditions on the separatrix $\mathcal{M}_0$ into $\mathcal{M}_a$ and $\mathcal{M}_b$, the basins of attraction of the two critical solutions. We can find $\mathcal{M}_1$ by following solutions from a two parameter family of initial conditions $\rho_0(r, p, q)$ and we expect that this manifold is given my the stable manifold of $h_2(r)$ and/or the stable manifold corresponding to the blowup profile $H_3(\eta)$.

The correspondence between the steady solutions and the self similar solutions with parabolic scaling and the fact that the dimensionalities of the unstable manifolds match suggests the possibility of constructing a global phase portrait for this system that encompasses both the smooth and the non smooth (blowup) behavior of the system.

The critical solution whose stable manifold separates the stable self-similar and non self-similar blowup corresponds to the profile H_*. Using the bisection method repeatedly, we can carefully choose the initial data to be close to the separatrix between the basins of the self similar blowup and non self similar blowup. In this way, we numerically follow the solutions to Eq. (8) along the unstable orbit. In this case, the unstable orbit is also a blowing up solutions and our solutions follow the unstable orbit until very close to the time of blowup. These solutions eventually diverge from the critical solution but during this transient period, the density blows up on the self-similar scales and converges in the similarity variables towards a profile which is in very good agreement with the solution Π_* of Eq. (64) satisfying the boundary condition (63) and corresponding to the unstable solution H_*.

Fig. 17 shows numerically obtained solutions of Eq. (8) in $d = 3$ for an initial condition that is very close to the separatrix between the self similar blowup and non self similar blowup. The density is normalized by the inverse of the time to blowup and the radial coordinate is scaled by the square root of the time to blowup. The function Π_* is also shown for comparison. Note that the scaled density profiles start to diverge away from the curve given by Π_* for $t \sim T - 1.12 \times 10^{-22}$ because the solution given by Π_* is unstable and it only occurs in a transient phase for very carefully chosen initial data.

7. Conclusions

We study the Keller-Segel system which describes the evolution of a compressible active scalar that diffuses and is advected by the gradient it generates. We have discussed the existence, stability and correspondence of steady solutions and parabolic self-similar blow up solutions for the Keller-Segel system as a function of space dimension. We find that the system has rich dynamics with multiple steady states, multiple blowing up self-similar solutions with parabolic scaling and also solutions that lead to blowup on scales different from the parabolic scaling. Generalizing the previously known results for the case $d = 2$, we show that on a bounded domain Ω_R, there exists a critical mass such that there are no steady solutions with a mass larger than the critical mass for all $d > 2$. We also prove results on the dimension of the unstable manifolds of the various steady

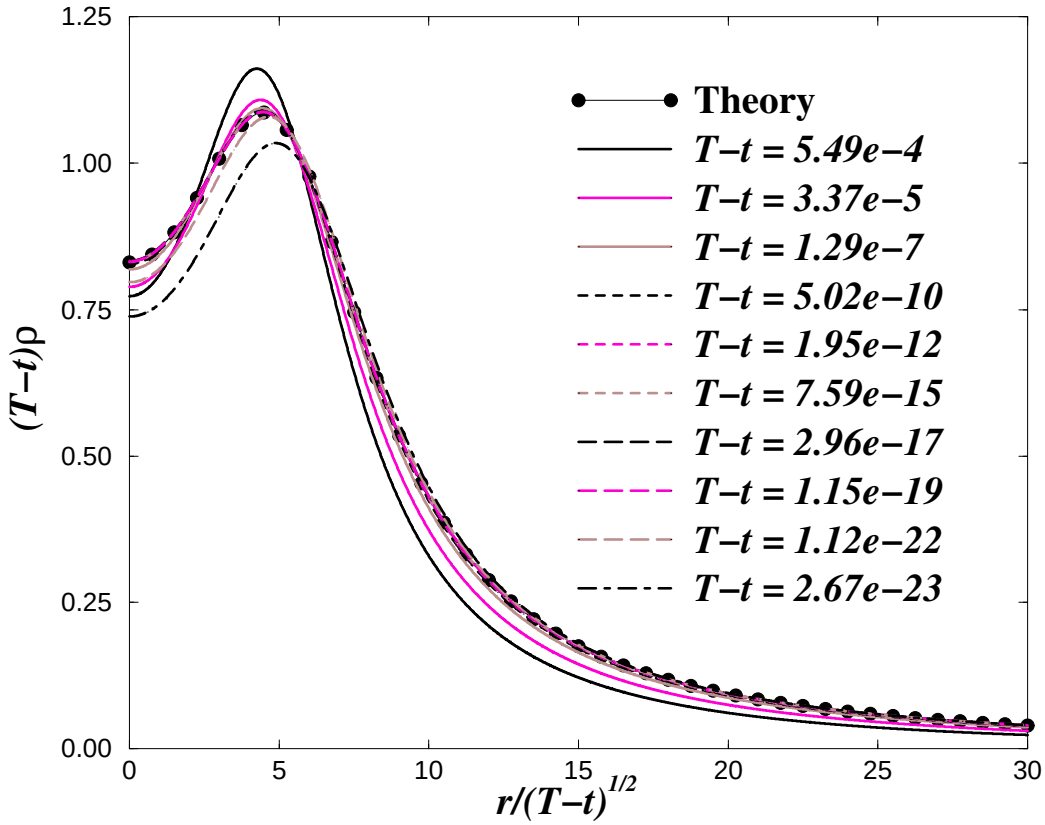

Figure 17. Fig. 5: Blowup profile in $d = 3$ for an initial condition that is close to the separatrix between the self similar blowup and the non self similar blowup. The blowup time T is the blowup time for the solution. T is very close to the blowup time of the unstable solution that it tracks in the transient phase because the initial condition is chosen very close to the separatrix. The theoretical curve is the density profile for the solution Π_* and there are no parameters that are fit.

solutions and the self-similar blowup profiles. The presence of multiple steady solutions and blowing up solutions with unstable manifolds of matching dimension suggests the possibility of constructing a global phase portrait that encompasses both the smooth behavior and the singularity formation. We present some results on the structure of the phase space for this system by exhibiting critical solutions whose stable manifolds are boundaries between qualitatively different dynamics.

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Appendix A.

We will now prove the result in Theorem 6 that was stated in Sec. 5.1.

Theorem 6. *For $2 < d < 10$, there exist a countable number of solutions to the equation*

$$\partial_\eta^2 H + \frac{d-3+2H}{\eta}\partial_\eta H + \frac{2(d-2)H}{\eta^2}(H-1) - \frac{\eta}{2}\partial_\eta H = 0.$$

satisfying the boundary conditions $H(\eta)/\eta^2$ is bounded as $\eta \to 0$ and $\eta\partial_\eta H \to 0$ as $\eta \to \infty$. These solutions also have the properties that $H(\eta) \geq 0$ and $G(\eta) = (1/\eta^{d-3})\partial_\eta(\eta^{d-2}H(\eta)) \geq 0$ for all η. Further, given an non-negative integer N, there exists a solution H_N such that the solution $H_N(\eta)$ intersects the line $H = 1$ precisely N times.

Note that Eq. (66) differs from Eq. (14) only in the presence of the term $-\frac{\eta}{2}\partial_\eta H$. Because of this term, Eq. (66) does not have a scale-invariance and we cannot analyze this equation by converting it to an autonomous system (*cf.* Sec. 3.3). However, the additional term in this equation is a small perturbation, if η is small, and the solutions of this equation will therefore be "close" to the solutions of Eq. (14) for small η. For large η, the additional term becomes significant and we need to analyze the system to deduce the properties of the solutions.

Although Eq. (66) is not scale-invariant, it is useful to define the quantity $\xi = \log(\eta)$ analogous to the earlier definition of $z = \log(r)$ (See Sec. 3.3). We will also define the quantity $J = H_\xi$. Then,

$$G = (d-2)H + J. \tag{A.1}$$

Eq. (66) can be written as the system of equations

$$\begin{aligned} H_\xi &= J, \\ J_\xi &= -\left[(d-2) + 2(H-1) - \frac{e^{2\xi}}{2}\right] J - 2(d-2)H(H-1). \end{aligned} \tag{A.2}$$

We will use this form of the equations to analyze the behavior for large ξ. For ξ negative, it is useful to rewrite the system in terms of G :

$$\begin{aligned} H_\xi &= G - (d-2)H, \\ G_\xi &= -2(H-1)G + \frac{e^{2\xi}}{2}[G - (d-2)H]. \end{aligned} \tag{A.3}$$

We choose ξ_0 such that

(i) $\xi_0 > 4(d-2) > 0$. and

(ii) $\exp[2\xi]/2 \geq \xi^2 + d$ for all $\xi \geq \xi_0$.

We will define the set $\mathcal{S} = \{H \geq 0, G \geq 0\}$. For a solution $(H(\xi), G(\xi))$ to be a self-similar blowup profile for initial data with a non-negative density $\rho_0(r)$, we need that $(H(\xi), G(\xi)) \in \mathcal{S}$ for $\xi \in (-\infty, \infty)$. It is also convenient to define $u = H - 1$ so that the number of intersections of the graph of H with the line $H = 1$ is the same as the number of zeros of u. With these definitions, we have

Lemma A.1. *Given $U_\infty \in [-1, 1]$, there exists a solution of the system in (A.2) for $\xi \in [\xi_0, \infty)$ such that $\lim_{\xi\to\infty} H = 1 + U_\infty$ and $\lim_{\xi\to\infty} J = 0$. The functions u and J are bounded and they do not change sign for $\xi \in [\xi_0, \infty)$. Further, $|u(\xi)| + |J(\xi)| \leq (1 + 1/\xi_0)|U_\infty|$ for $\xi \in [\xi_0, \infty)$.*

Proof. We can rewrite (A.2) in terms of u

$$\begin{aligned} u_\xi &= J, \\ J_\xi &= -\left[(d-2) + 2u - \frac{e^{2\xi}}{2}\right] J - 2(d-2)u(1+u). \end{aligned} \tag{A.4}$$

Assume that $U_\infty \in [-1, 1)$ with $U_\infty \neq 0$. Choose $N > \xi_0$. The functions on the RHS in (A.4) satisfy a Lipschitz condition on the domain $|u| \leq 2, \xi \leq N$ for any $N < \infty$. Consequently, for there exists an $a_N < N$ such that (A.2) has a unique solution $(u_N(\xi), J_N(\xi))$ on the interval $(a_N, N]$ with $u_N(N) = U_\infty$, $J_N(N) = 0$ and $|u_N| \leq 2$. Also, there exists an interval $[b_N, N]$ such that $u_N > -1$ and u_N does not change sign over this interval.

From (A.4) we have,

$$u_{\xi\xi} + \left[(d-2) + 2u - \frac{e^{2\xi}}{2}\right] u_\xi = -2(d-2)u(1+u). \tag{A.5}$$

If $U_\infty > 0$, u_N is non-negative in the interval $[b_N, N]$. Consequently, u_N satisfies a minimum principle [17], so that the minimum value of u_N can only occur at either $\xi = b_N$ or $\xi = N$. Further, if the minimum occurs at b_N, we should have $J_N(b_N) > 0$ and if the minimum occurs at N, we should have $J_N(N) < 0$ with the inequalities being strict. However, $J_N(N) = 0$, so that the minimum cannot be at $\xi = N$. Because u_N cannot have any local minima in $[b_N, N]$, it follows that $0 \leq u_N(\xi) \leq U_\infty \leq 1$ and $J_N(\xi) \geq 0$ for $\xi \in [b_N, N]$. Similarly, if $U_\infty < 0$, u_N satisfies a maximum principle and we have that $0 \geq u_N(\xi) \geq U_\infty > -1$ and $J_N(\xi) \leq 0$ for $\xi \in [b_N, N]$.

Let the function v be u_N if $U_\infty > 0$ and $-u_N$ if $U_\infty < 0$. Then, from the above results and (A.4), we have the differential inequality

$$L[v] \equiv v_{\xi\xi} + \left[d - \frac{e^{2\xi}}{2}\right] v_\xi + 4(d-2)v \geq 0. \tag{A.6}$$

Let $w(\xi) = 1 - 4(d-2)/\xi$. Then, $w > 0$ for $\xi \in [\xi_0, N]$. A direct computation yields

$$w_{\xi\xi} - (\xi^2 - 4(d-2)\xi - 2)w_\xi + 4(d-2)w = 0.$$

Using the definition of ξ_0 and using $w_\xi \geq 0$, we see that $L[w] \leq 0$ for $\xi \in [\xi_0, N]$. The generalized maximum principle [17] implies that $z = v/w$ satisfies a maximum principle on $[\xi_0, N]$. Since $z_\xi(N) = (v_\xi w - v w_\xi)/w^2 < 0$, the maximum of z cannot occur at $\xi = N$. Consequently, r does not change sign over the interval $[\xi_0, N]$. Because $w > 0$ over this interval, it follows that u_N and J_N do not change sign on $[\xi_0, N]$.

If $U_\infty = 0$ of $U_\infty = -1$, a direct computation yields $u_N(\xi) = U_\infty$, $J_N(\xi) = 0$ is a solution for $\xi \in [\xi_0, N]$ and for these solutions also it is true that u_N and J_N do not change sign on $[\xi_0, N]$.

Defining $u_N = U_\infty$ and $J_N = 0$ for $\xi > N$, we can extend the definition of u_N and J_N to the set $[\xi_0, \infty)$ such that u_N and J_N are continuous on this set, and $(u_N)_\xi = J_N$. Because (u_N, J_N) is a solution of (A.4) for $\xi \in [\xi_0, N]$, we can integrate the second equation in the system to obtain, for all $\xi \in [\xi_0, N]$,

$$J_N(\xi) = \int_\xi^N B_N(\xi')\Phi(\xi, \xi')d\xi', \tag{A.7}$$

with

$$A_N(\xi'') = \frac{e^{2\xi''}}{2} - (d-2) - 2u_N(\xi'')$$
$$B_N(\xi) = 2(d-2)u_N(\xi)(1 + u_N(\xi))$$

and

$$\Phi_N(\xi, \xi') = \exp\left[-\int_\xi^{\xi'} A_N(\xi'')d\xi''\right].$$

With these definitions, it is clear that B_N is bounded if u_N is bounded. We can bound Φ_N because $A_N(\xi'') \geq (\xi'')^2$ in the range of integration. We have,

$$\begin{aligned} |\Phi_N(\xi, \xi')| &\leq \exp\left[-\int_\xi^{\xi'} (\xi'')^2 d\xi''\right] \\ &\leq \exp\left[-\xi^2(\xi' - \xi)\right]. \end{aligned} \tag{A.8}$$

Using $|u_N| \leq |U_\infty| \leq 1$

$$\begin{aligned} |J_N(\xi)| &\leq \int_\xi^N 2(d-2)|u_N(\xi')|(1 + |u_N(\xi')|)|\Phi(\xi, \xi')|d\xi' \\ &\leq 4(d-2)|U_\infty| \int_\xi^\infty \exp\left[-\xi^2(\xi' - \xi)\right] d\xi' \\ &= \frac{4(d-2)|U_\infty|}{\xi^2}. \end{aligned} \tag{A.9}$$

For $\xi \geq N$, we have $J_N(\xi) = 0$. Consequently, we have $|(u_N)_\xi| = |J_N| \leq 4(d-2)/\xi_0^2$ for $\xi \in [\xi_0, \infty)$.

By the previous result, the functions u_N are equicontinuous on this set and each u_N has a bounded range. Therefore, by the Arzelá-Ascoli theorem (*cf.* Royden [18]), we have that

(i) There is a subsequence of u_{N_i} that converges pointwise to a function u.

(ii) The pointwise limit u is a continuous function.

(iii) The convergence is uniform on every compact subset of $[\xi_0, \infty)$.

The convergence is actually uniform on $[\xi_0, \infty)$. Given any $\epsilon > 0$, choose $\xi_m = 4(d-2)/\epsilon$. By the Arzelá-Ascoli theorem, there exists a $n(\epsilon)$ such that for all $p, q > n(\epsilon)$, $|u_{N_p}(\xi) - u_{N_q}(\xi)| \leq \epsilon$ on the interval $[\xi_0, \xi_m]$. By (A.9) and the fact that $J_N \geq 0$, we have that

$$|u_N(\xi) - u_M(\xi)| \leq 4(d-2)/\xi \tag{A.10}$$

for all N, M. Therefore, from our definition of ξ_m, we obtain that $|u_{N_p}(\xi) - u_{N_q}(\xi)| \leq \epsilon$, and the convergence is uniform for all $\xi \geq \xi_0$. Without loss of generality, we can assume that $n(\epsilon) \to \infty$ as $\epsilon \to 0$.

We will now prove that u, the limit of the subsequence gives a solution of (A.4) with $\lim_{\xi\to\infty} u = U_\infty$ and $\lim_{\xi\to\infty} J = 0$. Define $e(\xi)$ by

$$e(\xi) = u(\xi) - \left(U_\infty - \int_\xi^\infty \int_{\xi'}^\infty B(\xi'')\Phi(\xi', \xi'')d\xi''d\xi' \right),$$

where the definitions of B and Φ are analogous to the definitions of B_N and Φ_N with u in place of u_N. It is clear that u is a solution of (A.4) with the required properties iff $e = 0$.

Choose $\epsilon > 0$. Then, by the uniform convergence of the subsequence, there exists a $N(\epsilon)$ such that $|u - u_{N(\epsilon)}| \leq \epsilon$. If $\xi > N(\epsilon)$, taking the limit $M \to \infty$ with $N = N(\epsilon)$, we get $|u(\xi) - U_\infty| \leq 4(d-2)/\xi$. The double integral can be bounded by observing that $B(\xi)$ is bounded and using (A.8), from which we have

$$e(\xi) \leq C/\xi \leq C/N(\epsilon).$$

Here and henceforth we will use C to denote an arbitrary constant, possibly changing from one line to the next.

We now consider the case $\xi \leq N(\epsilon)$. Eq. (A.7) yields

$$u_N(\xi) = U_\infty - \int_\xi^N \int_{\xi'}^N B_N(\xi'')\Phi_N(\xi', \xi'')d\xi''d\xi'.$$

Using this equation to estimate e, we have

$$\begin{aligned} |e(\xi)| \leq & \left| \int_\xi^N \int_N^\infty B(\xi'')\Phi(\xi', \xi'')d\xi''d\xi' \right| \\ & + \left| \int_N^\infty \int_{\xi'}^\infty B(\xi'')\Phi(\xi', \xi'')d\xi''d\xi' \right| \end{aligned}$$

$$+\left|\int_{\xi}^{N}\int_{\xi'}^{N}\left[B(\xi'')\Phi(\xi',\xi'')-B_N(\xi'')\Phi_N(\xi',\xi'')\right]d\xi''d\xi'\right|$$
$$+\left|u(\xi)-u_N(\xi)\right|, \tag{A.11}$$

where $N=N(\epsilon)$. We will estimate each of these terms. We will repeatedly use the facts that B and B_N are bounded and that $|\Phi(\xi,\xi')|\le\exp(-\xi^2(\xi'-\xi))$. The first term gives

$$\left|\int_{\xi}^{N}\int_{N}^{\infty}B(\xi'')\Phi(\xi',\xi'')d\xi''d\xi'\right|\le C\int_{\xi}^{N}\frac{d\xi'}{N^2}\le\frac{C}{N}.$$

For the second term,

$$\left|\int_{N}^{\infty}\int_{\xi'}^{\infty}B(\xi'')\Phi(\xi',\xi'')d\xi''d\xi'\right|\le C\int_{N}^{\infty}\frac{d\xi'}{\xi'^2}\le\frac{C}{N}.$$

By the uniform convergence of the subsequence, we have $|u-u_N|\le\epsilon$. It follows that, $|B-B_N|\le C\epsilon$ because both u and u_N are bounded. Also,

$$\begin{aligned}|\Phi(\xi,\xi')-\Phi_N(\xi,\xi')| &= \left|\Phi(\xi,\xi')\left[1-\exp\left(\int_{\xi}^{\xi'}2(u_N(\xi'')-u(\xi''))d\xi''\right)\right]\right| \\ &\le \exp(-\xi^2(\xi'-\xi))(\exp(2\epsilon(\xi'-\xi))-1)\end{aligned} \tag{A.12}$$

Writing $B\Phi-B_N\Phi_N$ as $(B-B_N)\Phi+B_N(\Phi-\Phi_N)$ and using the above results, the estimate for the third term is given by

$$\left|\int_{\xi}^{N}\int_{\xi'}^{N}\left[B(\xi'')\Phi(\xi',\xi'')-B_N(\xi'')\Phi_N(\xi',\xi'')\right]d\xi''d\xi'\right|\le C\epsilon.$$

By the uniform convergence, the fourth term gives $|u-u_N|\le\epsilon$. Putting all these results together, we see that

$$|e(\xi)|\le C/N(\epsilon)+C\epsilon,$$

for all $\xi\ge\xi_0$. Since ϵ was chosen arbitrarily and $N(\epsilon)\to\infty$ as $\epsilon\to 0$, we see that $e(\xi)=0$. Therefore, the function u is a solution of (A.4) for $\xi\in[\xi_0,\infty)$ with $\lim_{\xi\to\infty}u=U_\infty$. Also, if $U_\infty\neq 0$, u has no zeros for $\xi\ge\xi_0$.

Combining (A.9) with the fact that $|u(\xi)|\le|U_\infty|$ for $\xi\in[\xi_0,\infty)$, we see that $|u(\xi)|+|J(\xi)|\le(1+1/\xi_0)|U_\infty|$ on this interval.

□

Lemma A.2. *With ξ_0 as defined above, $u(\xi_0)$ and $J(\xi_0)$ depend continuously on U_∞ for $U_\infty\in[-1,1]$.*

Proof. For any $U_\infty\in[-1,1]$, the previous lemma guarantees the existence of a solution $(u(\xi,U_\infty),J(\xi,U_\infty))$. Let

$$s=\frac{\partial u}{\partial U_\infty},K=\frac{\partial J}{\partial U_\infty}.$$

We can verify that, $\lim_{\xi\to\infty} s = 1$ and $\lim_{\xi\to\infty} K = 0$. Differentiating (A.4), we get

$$\begin{aligned} s_\xi &= K, \\ K_\xi &= -\left[(d-2) + 2u - \frac{e^{2\xi}}{2}\right] K - (2J + 2(d-2)(1+2u))s. \end{aligned} \tag{A.13}$$

From the second equation, we obtain

$$K = \int_\xi^\infty (2J + 2(d-2)(1+2u))s(\xi')\Phi(\xi,\xi')d\xi'$$

Let $S(\xi) = \max_{\xi'\geq\xi} |s(\xi')|$. The previous lemma shows that u and J are bounded. Using the bound on Φ (Eq. (A.8)), we have

$$|s_\xi| = |K(\xi)| \leq \frac{C}{\xi^2} S(\xi).$$

The Gronwall inequality yields $|s(\xi)| \leq s(\infty)\exp[C/\xi] = \exp[C/\xi]$ and the above result gives $|K(\xi)| \leq C/\xi^2 \exp[C/\xi]$. This proves that $u(\xi_0)$ and $J(\xi_0)$ are C^1 functions of U_∞. □

Lemma A.3. *Let $\mathcal{K} \subset \mathcal{S}$ be a compact set. If $(H_0, G_0) \in$ Interior$[\mathcal{K}] = \mathcal{K} - \partial\mathcal{K}$ and $\xi_1 < \infty$, there exists a smallest $\xi_2 < \xi_1$ and a unique solution $(H(\xi), G(\xi))$ of the system in (A.3) for $\xi \in (\xi_2, \xi_1)$ with $(H(\xi_1), G(\xi_1)) = (H_0, G_0)$ such that $(H(\xi), G(\xi)) \in \mathcal{K}$ for $\xi \in (\xi_2, \xi_1)$. If $\xi_2 = -\infty$ it follows that $\lim_{\xi\to\infty}(H(\xi), G(\xi))$ exists and if $\xi_2 > -\infty$, $(H(\xi_2), G(\xi_2)) \in \partial\mathcal{K}$. Also, if $\|(H(\xi_1), G(\xi_1)) - (1, d-2)\| > 0$, $\lim_{\xi\to\infty}(H(\xi), G(\xi)) = (0,0)$.*

Proof. The local existence of the solution and the extension to a maximal interval (ξ_2, ξ_1) follow from standard results on the existence and uniqueness of solutions for ODEs. In the case $\xi_2 = -\infty$, define the limit set α by

$$\alpha = \bigcap_{\xi\leq\xi_1} \overline{\bigcup_{\xi'\leq\xi} (H(\xi'), G(\xi'))}$$

The set α depends on (H_0, G_0) and on ξ_1 but we will suppress this dependence to keep the notation simple. By the compactness of $\mathcal{K}$, the set α is nonempty. Note that the difference between the equations in (A.3) and (30) is a term of the form

$$\frac{e^{2\xi}}{2}\left[G - (d-2)H\right],$$

and this term can be uniformly bounded by $Me^{2\xi}$ on $\mathcal{K}$ and this bound goes to zero asymptotically as $\xi \to -\infty$. A standard calculation shows that given $\epsilon > 0$, we can find a ξ_1 sufficiently small, so that a solution of (30) approximates the solution of (A.3) to within ϵ for $\xi' \leq \xi_1$ in the norm $|||(H_1, G_1) - (H_2, G_2)||| = \|H_1 - H_2\| + \|\log(G_1/G_2)\|$. Consequently, α is contained in the α-limit set of the flow in (30) which is a two

dimensional autonomous system, and it can only consist of critical points or limit cycles of the system in (30).

In Sec 3.3, we showed that the flow in (30) has no limit cycles. Consequently, α can only contain the isolated critical points $(0,0)$ and $(1,(d-2))$ or the point at ∞. $\alpha \subset \mathcal{K}$ so that the point at ∞ is ruled out. The solutions to the equations in (A.3) are continuous functions of ξ. It follows that, for a given (H_0, G_0) and ξ_1, the set α is connected so that the set α is either the point $(0,0)$ or the point $(1,(d-2))$. Therefore, if there exists a solution $(H(\xi), G(\xi))$ of the system in (A.3) for $\xi \in (-\infty, \xi_1)$ such that $(H(\xi), G(\xi)) \in \mathcal{K}$ for $\xi < \xi_1$, it follows that $\lim_{\xi\to\infty}(H(\xi), G(\xi))$ exists.

The point at $(1, d-2)$ is a stable (attracting) focus for the system in (30). Using the standard method, one can construct a positive definite Lyapunov functional $E(h,g)$ with $E(1,d-2) = 0$, such that for all (h,g) sufficiently close to but not equal to $(1,d-2)$, we have that

$$\frac{dE}{dz} < 0$$

on solutions of (30). We can check that for sufficiently negative ξ and for (H,G) sufficiently close to but not equal to $(1,d-2)$, we have that

$$\frac{dE(H,G)}{d\xi} < 0$$

on solutions of (A.3). Therefore, we cannot have that $\lim_{\xi\to-\infty}(H(\xi), G(\xi)) = (1, d-2)$. □

Corollary A.3.1. *If $\mathcal{K}$ is a compact subset of $\mathcal{S}$ not containing $(0,0)$, and (H_0, G_0) is an initial condition not equal to $(1,d-2)$, it follows that there is a finite $\xi_2 < \xi_1$ such that the solution is never in the set $\mathcal{K}$ for $\xi < \xi_2$.*

In order to simplify the notation, we will use boldface letters $\mathbf{x}$ and $\mathbf{y}$ to represent a point (H,G) and we will write (A.3) as

$$\frac{d\mathbf{x}}{d\xi} = \mathbf{F}(\mathbf{x}, \xi).$$

We will represent solutions of this equation by $\mathbf{x}(\xi)$. Also, for any compact set $\mathcal{K}$ in the phase space, we will use $\mathbf{n}$ to represent the outward normal at a point in $\partial\mathcal{K}$. With these conventions, we have the following result.

Lemma A.4. *Given $\mathbf{y} \in \mathcal{S}$ with $\mathbf{y} \neq (1,d-2)$ and $\xi_1 < \infty$, there exist $G_2(\mathbf{y}) \in [0,\infty)$ and $\xi_2(\mathbf{y}) \leq \xi_1$ such that there exists a unique solution $\mathbf{x}(\xi)$ of (A.3) with $\mathbf{x}(\xi_1) = \mathbf{y}$, $\mathbf{x}(\xi) \in \mathcal{S}$ for $\xi \in (\xi_2(\mathbf{y}), \xi_1]$, $\lim_{\xi\to\xi_2(\mathbf{y})} H(\xi) = 0$, and $\lim_{\xi\to\xi_2(\mathbf{y})} G(\xi) = G_2(\mathbf{y})$.*

Note that $G_2(\mathbf{y})$ and $\xi_2(\mathbf{y})$ also depend on ξ_1 but we are suppressing the argument ξ_1 for notational convenience.

Proof. Figure A1 shows the phase space of the system (A.3). We are given $\xi_1 < \infty$. Define $\beta = \exp(2\xi_1)/2$. The segment AB is given by $0 \leq H \leq 1$ and $G = (d-1)e^{2+\beta}$.

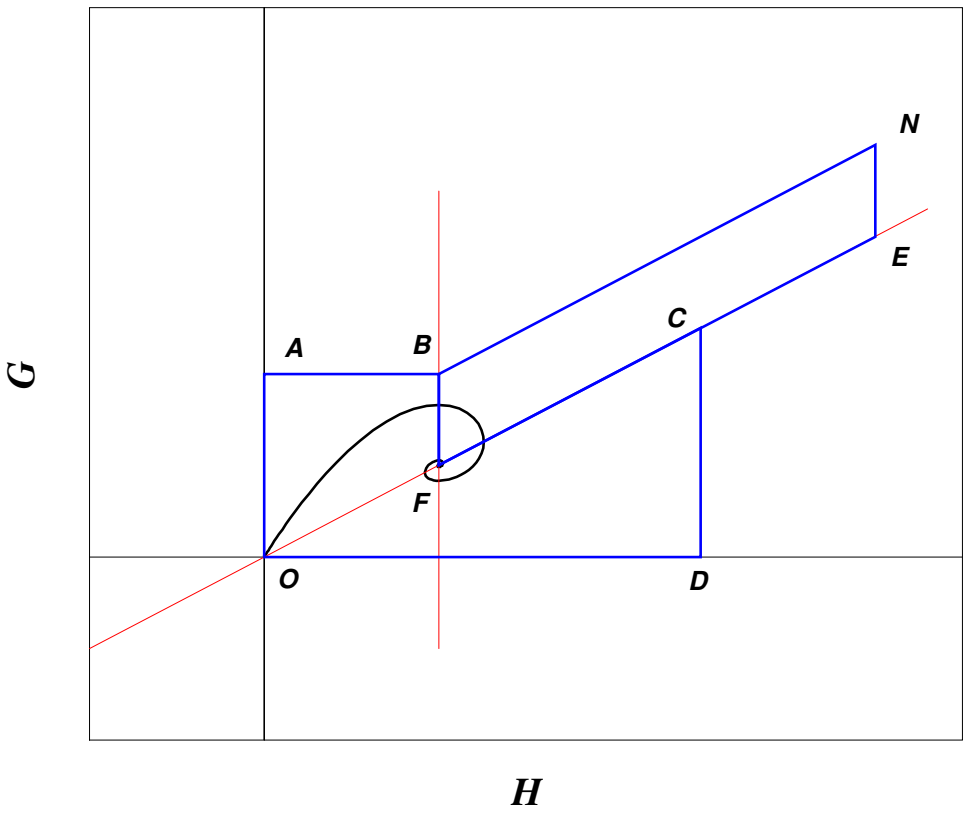

Figure A1. Phase space (H and G) of the system in Eq. (A.3). A solution with $\lim_{\xi\to-\infty}(H,G)=(0,0)$ is also shown.

F if the point $(1,d-2)$ and the segment CD is given by $H = d/2+1$ and $0 \le G \le (d-2)(d/2+1)$. We will denote the compact set $OABFCD$ by $\mathcal{K}_1$. Let $\mathbf{y} = \mathbf{x}(\xi_i)$ belong to this set. Then, there exists a smallest ξ_f such that the solution of (A.3) has $\mathbf{x}(\xi) \in \mathcal{K}_1$ for $\xi \in (\xi_f, \xi_i]$. If $\xi_f = -\infty$, Lemma A.3 implies the result in Lemma A.4 with $G_2(\mathbf{y}) = 0$ and $\xi_2(\mathbf{y}) = -\infty$. If $\xi_f > -\infty$, we have that $\mathbf{x}(\xi_f) \in \partial\mathcal{K}_1$. On AB and OD, $\mathbf{F}\cdot\mathbf{n} \ge 0$. Therefore, $\mathbf{x}(\xi_f) \in OA \cup FC \cup CD$. If $\mathbf{x}(\xi_f) \in OA$, we have our result with $G_2(\mathbf{y}) > 0$ and $\xi_2(\mathbf{y}) > -\infty$.

Assume that $\mathbf{x}(\xi_i) \in \mathcal{U}_1 = \{H \ge d/2+1, 0 < G < (d-2)H\}$. On CD, $H_\xi < 0$. Therefore, there exists a smallest ξ_f such that $\mathbf{x}(\xi) \in \mathcal{U}_1$ for $\xi \in [\xi_f, \xi_i]$. In this region, $G_\xi \le -dG$ so that $G(\xi) \ge G(\xi_i)\exp[d(\xi_i - \xi)]$ and $H_\xi \ge -(d-2)H$ so that $H(\xi) \le H(\xi_i)\exp[(d-2)(\xi_i - \xi)]$ for $\xi \le \xi_i$. Consequently, there exists a $\xi_f > -\infty$ such that $G(\xi_f) = (d-2)H(\xi_f) < \infty$. Combining this result with the result in the previous paragraph, we see that for any $\mathbf{x}(\xi_i) \in \mathcal{K}_1 \cup \mathcal{U}_1$, either Lemma A.4 is true or there exists

a $H_2 < \infty$ and a $\xi_f > -\infty$ with $\mathbf{x}(\xi_f) \in FE$ where E is the point $(H_2, (d-2)H_2)$.

Now assume that $\mathbf{x}(\xi_i) \in \mathcal{U}_2 = \{H \geq 1, 0 \leq G - (d-2)H < (d-1)(e^{2+\beta} - 1) + 1\}$. Choose a $H_2 < \infty$ such that $H(\xi_i) \leq H_2$ and let EN be the segment $\{H = H_2, 0 \leq G - (d-2)H \leq (d-1)(e^{2+\beta} - 1) + 1\}$. We will let $\mathcal{K}_2$ denote the compact set $BFEN$. By the previous Lemma and the fact that $(0,0) \notin \mathcal{K}_2$, it follows that there exists a $\xi_f > -\infty$ such that $\mathbf{x}(\xi_f) \in \partial\mathcal{K}_2$. On $FE \cup EN, \mathbf{F} \cdot \mathbf{n} \geq 0$. Therefore $\mathbf{x}(\xi_f) \in FB \cup BN$.

We have already considered the case $\mathbf{x}(\xi_i) \in FB \subset \mathcal{K}_1$. We now consider the case $\mathbf{x}(\xi_i) \in \mathcal{U}_3 = \{H \geq 1, G \geq (d-2)H + (d-1)(e^{2+\beta} - 1) + 1\}$. In this set, $H_\xi \geq (d-1)(e^{2+\beta} - 1) + 1$ so that $H(\xi) \leq H(\xi_i) - [(d-1)(e^{2+\beta} - 1) + 1](\xi_i - \xi)$ and $G_\xi \geq -2(H-1)G \geq -2(H(\xi_i))G$ so that $G(\xi) \leq G(\xi_i) \exp[2H(\xi_i)(\xi_i - \xi)$. It therefore follows that there exists a $\xi_f > -\infty$ with $H(\xi_f) = 1$ and $G(\xi_f) < \infty$.

Finally, let us assume that $\mathbf{x}(\xi_i) \in \mathcal{U}_4 = \{0 \leq H \leq 1, G \geq (d-1)e^{2+\beta}\}$. In the region $0 \leq H \leq 1, G \geq (d-1)$, we have $H_\xi \geq 1$ and $G_\xi \geq 0$ so that there exists a $\xi_f > \xi_i - 1$ when the solution $\mathbf{x}(\xi_f)$ should either have $H(\xi_f) = 0$ or $G(\xi_f) = (d-1)$. For $0 \leq H \leq 1$, $G_\xi \leq (2+\beta)G$ so that $G(\xi_f) \geq G(\xi_i) \exp[(2+\beta)(\xi_i - \xi_f)]$. Therefore, for an initial condition in $\mathcal{U}_4$, $G(\xi_f) \geq (d-1)$. It follows that for $\mathbf{y} \in \mathcal{U}_4$, Lemma A.4 is true since for any $\xi_i \leq \xi_1$, there exists a $\xi_2(\mathbf{y}) = \xi_f > -\infty$ with $H(\xi_f) = 0$ and $G(\xi_f) = G_2(\mathbf{y}) \geq (d-1)$.

We have shown that given any $\xi_i < \infty$ and $\mathbf{x}(\xi_i) \in \mathcal{K}_1 \cup \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{U}_4 = \{H \geq 0, G \geq 0\}$ there exists a $\xi_f < \xi_i$ (possibly $-\infty$) with either $\lim_{\xi \to \xi_f} H(\xi_f) = 0$ or $H(\xi_f) \in BF$. In the latter case, $\xi_f > -\infty$.

We will now prove Lemma A.4 by contradiction. Assume that the claim is false. Then repeatedly applying the result in the previous paragraph will give sequences $\xi_n > -\infty$ and $\mathbf{x}_n = \mathbf{x}(\xi_n) \in BF$ for $n = 1, 2, \ldots$. The set BF is closed and there exists an accumulation point $\mathbf{x}^* \in BF$ for the sequence. This gives the required contradiction with the result in the corollary to Lemma A.3.

Combining Lemma A.3 and Lemma A.4, we see that if $\mathbf{y}$ and ξ_1 are such that $G_2(\mathbf{y}) > 0$, we have $\xi_2(\mathbf{y}) > -\infty$. □

Corollary A.4.1. *Given $\xi_1 < \infty$ and $\mathbf{x}_0 \in \mathcal{S}$, the solution of (A.3) to the initial value problem $\mathbf{x}(\xi) = \mathbf{x}_0$ has a finite number of intersections with the line $H = 1$ for $\xi \leq \xi_1$. This is true, since the solution would otherwise intersect the compact set BF an infinite number of times and we will contradict the corollary to Lemma A.3.*

Lemma A.5. *$\mathbf{x}(\xi)$ is a solution of (A.3). If ξ_1 and $\mathbf{y}_0 = \mathbf{x}(\xi_1)$ are such that $\xi_2(\mathbf{y}_0) = \xi_2^* > -\infty$ so that $G_2(\mathbf{y}) = G_2^* > 0$, then there exists a connected open neighborhood $\mathcal{U}$ of $\mathbf{y}_0$ such that for each $\mathbf{y} \in \mathcal{U}$, $\xi_2(\mathbf{y}) > -\infty$. Further, the map $\mathbf{y} \to (\xi_2(\mathbf{y}), G_2(\mathbf{y}))$ is C^1 on $\mathcal{U}$, and if $H_0 \neq 1$, we can always choose $\mathcal{U}$ such that the number of intersections of the solutions $\mathbf{x}(\xi)$ for $\xi \in [\xi_2(\mathbf{x}), \xi_1]$ of (A.3) with the line $H = 1$ is the same for all initial conditions $\mathbf{x}(\xi_1) \in \mathcal{U}$.*

Proof. Given the solution $\mathbf{x}(\xi)$ with $\mathbf{x}(\xi_1) = \mathbf{y}_0$, we can choose a compact set $\mathcal{K}$ such that the solution $\mathbf{x}(\xi)$ is uniformly bounded away from $\partial\mathcal{K}$. On this compact set $\mathcal{K}$ and

for $\xi_2^* \le \xi \le \xi_1$, the real part of the eigenvalues of the Jacobian Matrix

$$\mathbf{M}_{ij}(\xi) = \frac{\partial \mathbf{F}_i(\mathbf{x}, \xi)}{\partial \mathbf{x}_j}$$

are uniformly bounded below by a constant $-C$. Therefore, if $\mathbf{x}_1(\xi)$ and $\mathbf{x_2}(\xi)$ are two solutions of (A.3), we have that

$$\|\mathbf{x}_1 - \mathbf{x}_2\|(\xi) \le \|\mathbf{x}_1 - \mathbf{x}_2\|(\xi_1) e^{C(\xi_1 - \xi)}.$$

for $\xi \le \xi_1$, so that $\mathbf{x}(\xi_2^*)$ is a C^1 function of $\mathbf{x}(\xi_1)$. Also, if $G(\xi_2^*) > 0$, we have a neighborhood $\mathcal{V}$ of $(0, G)$ where H_ξ is bounded away from zero by a constant C_1 and G_ξ is bounded by a constant C_2 for all ξ. It follows that we can choose the open set $\mathcal{U}$ such that

$$|\xi_2(\mathbf{y}) - \xi_2^*| \le \frac{\|\mathbf{y} - \mathbf{y}_0\| e^{C(\xi_1 - \xi_2^*)}}{C_1}$$

and

$$|G_2(\mathbf{y}) - G_2^*| \le \|\mathbf{y} - \mathbf{y}_0\| e^{C(\xi_1 - \xi_2^*)} \left(1 + \frac{C_2}{C_1}\right)$$

are smaller than any given $\epsilon > 0$ demonstrating the continuity of the map $\mathbf{y} \to (\xi_2(\mathbf{y}), G_2(\mathbf{y}))$ on a sufficiently small neighborhood of $\mathbf{y}_0$. Let this neighborhood be $\mathcal{U}$. Without loss of generality, we can choose $\mathcal{U}$ so that it is connected. The continuity of $\xi_2(\mathbf{y})$ on $\mathcal{U}$ implies that $\xi_2(\mathbf{y}) > -\infty$ for $\mathbf{y} \in \mathcal{U}$.

If the initial condition has $H_0 \neq 1$, we have $\mathbf{y}_0 \neq (1, d-2)$. We can choose $\mathcal{U}$ such that $(1, d-2) \notin \overline{\mathcal{U}}$ and the line $H = 1$ does not intersect $\mathcal{U}$. As we showed earlier, the existence of a Lyapunov functional for (30) implies that the solutions $\mathbf{x}(\xi)$ of (A.3) are bounded away from $(1, d-2)$ for every initial condition $\mathbf{y} \in \mathcal{U}$, so that on every intersection with the line $H = 1$, we have H_ξ is uniformly bounded away from zero. It follows that the location ξ_i of the i-th intersection is a continuous function of the initial condition $\mathbf{y}$. $\mathcal{U}$ can be chosen sufficiently small so that there are no intersections with $H = 1$ in the interval $[\xi_2(\mathbf{y}), \xi_2^*]$ since $H(\xi_2(\mathbf{y})) = 0$ and H_ξ is bounded on $\mathcal{K}$. If the number of intersections in $[\xi_2^*, \xi_1]$ changes as we vary $\mathbf{y}$ over $\mathcal{U}$, continuity and the connectedness of $\mathcal{U}$ imply the existence of an initial condition with $H(\xi_1) = 1$ with contradicts the fact that $\mathcal{U}$ does not intersect the line $H = 1$. □

Lemma A.6. *There exists a $\xi_m > -\infty$ and a neighborhood $\mathcal{W}$ of $(0, 0)$ such that every solution of the initial value problem $\mathbf{x}(\xi_m) = \mathbf{y} \in \mathcal{W}$ has either no intersections with the line $H = 1$ or precisely two intersections with the line $H = 1$ for $\xi \in (\xi_2(\mathbf{y}), \xi_m]$.*

Proof. Let $\mathcal{W} = \{0 \le H < a, 0 \le G < (d-2)a\}$, and $\gamma = \exp(2\xi_m)/2$, where $1 > a > 0$ and $\xi_m > -\infty$ are constants we will specify later. $\overline{\mathcal{W}}$ is compact. Using the same arguments as in Lemma A.4, we see that there exists a $\xi_n < \xi_m$ where the solution does not leave $\mathcal{W}$ for $\xi \in (\xi_n, \xi_m]$ and either $\lim_{\xi \to \xi_n} H(\xi) = 0$ in which case we are done, or $H(\xi_n) = a$ and $0 < G(\xi_n) \le (d-2)a$.

This solution can be extended to the set $[\xi_p, \xi_m]$ where $H(\xi_p) = 1$. In the region $0 \le H \le 1$, we have $H_\xi = G-(d-2)H$ and $G_\xi \ge \gamma[G-(d-2)H]$ so that $[\gamma H - G]_\xi \le 0$. This yields the inequality

$$G(\xi_p) \le \gamma(H(\xi_p) - H(\xi_n)) + G(\xi_n) \le \gamma + a. \tag{A.14}$$

so that by making a and γ sufficiently small, we can make $G(\xi_p)$ smaller than any given $\epsilon > 0$.

From the arguments used in proving Lemma A.4, we see that there exists a $\xi_q > -\infty$ where $G(\xi_p) = (d-2)H(\xi_p) > (d-2)$. Let $H_p < \infty$ denote a constant that we will choose below. In the region $H \le H_p$, we have that $G_\xi \ge -2H_pG - \gamma(d-2)H_p$. This gives the inequality

$$G(\xi) \le G(\xi_p)\exp[2H_p(\xi_p - \xi)] + \gamma(d-2)(\exp[2H_p(\xi_p - \xi)] - 1).$$

for $\xi < \xi_p$. Combining this with (A.14), we see that we can choose a and γ sufficiently small so that for $\xi \in [\xi_p - H_p, \xi_p]$, we have $G(\xi) < (d-1)$ *i.e*,

$$a\exp[2H_p^2] + \gamma(d-2)(2\exp[2H_p^2] - 1) < (d-1) \tag{A.15}$$

We also have $H_\xi = G-(d-2)H$, so that $H(\xi_p - H_p) \ge H(\xi_p) + ((d-2)H - G)(H_p) \ge H_p$. Consequently, by continuity, there exists a ξ_r with $H(\xi_r) = H_p$ and $G(\xi_r) < d-1$. Since $H_\xi < 0$ for $G \le (d-2)H$, it follows that at $\xi = \xi_q < \xi_r$, $H \ge H_p$ and $G = (d-2)H_p$.

In the region $G \ge (d-2)H$, $H \ge 1 + \gamma/2$, we have $H_\xi > 0$ and $G_\xi < 0$. Using Lemma A.4 we see that there exists a $\xi_s > -\infty$ with $H(\xi_s) = 1 + \gamma/2$ and $G(\xi_s) \ge (d-2)H_p$. Also, there exists a $\xi_t > -\infty$ with $H(\xi_t) = 1$. In the region $1 \le H \le 1 + \gamma/2$, we have $G_\xi \le \gamma G$ so that

$$G(\xi) \ge (d-2)H_p\exp[\gamma(\xi_s - \xi)]$$

Choose H_p such that

$$H_p \ge \frac{d-1}{d-2}\exp[3+\beta] + \frac{2}{d-2}, \tag{A.16}$$

where $\beta = \exp[2\xi_0]/2$ and $\xi_0 > 4(d-2)$ is as defined for Lemma A.1.

Without loss of generality, we can take $\gamma < 1$. Using $H_\xi = G - (d-2)H$ so that $H(\xi) \le H(\xi_s) - (\xi_s - \xi)$ so that $\xi_t \in [\xi_s - 1, \xi_s]$. We therefore have that

$$G(\xi_t) \ge (d-1)\exp[2+\beta].$$

It is clear that we can choose a and ξ_m such that (A.15) is satisfied with H_p given by (A.16), $\xi_m < \xi_0$ and $\gamma < 1$. Therefore $\xi_t < \xi_0$ and the arguments in Lemma A.4 show that there are no further intersections with the line $H = 1$ in $(\xi_2(\mathbf{y}), \xi_t]$. □

Lemma A.7. *Given any integer N and $\xi_1 < \infty$, there exists a δ such that $0 < \|\mathbf{y} - (1, d-2)\| < \delta$ implies that the solution to (A.3) with $\mathbf{x}(\xi_1) = \mathbf{y}$ has at least N intersections with the line $H = 1$ for $\xi \in (\xi_2(\mathbf{y}), \xi_1]$.*

Proof. We define r and θ so that $u = r\cos(\theta)$ and $J = r\sin(\theta)$, $r \geq 0$. From (A.2), we get, for sufficiently small r and for $\xi \leq \xi_1 < \infty$

$$\frac{r_\xi}{r} = \frac{1}{2}\frac{uu_\xi + JJ_\xi}{u^2 + J^2} \geq -A - Br, \tag{A.17}$$

for $r > 0$, with A and B are non-negative bounded constants. Choose $r_0 = B/(2A)$. Then, we have that, for $r \leq r_0$, $r_\xi \geq -3Ar/2$. Similarly, for θ we have

$$\begin{aligned}\theta_\xi &= \frac{uJ_\xi - Ju_\xi}{u^2 + J^2} \\ &= -\frac{J^2 + 2(d-2)u^2 + (d-2)uJ}{u^2 + J^2} + \frac{e^{2\xi}uJ}{2(u^2 + J^2)} + T(r,\theta),\end{aligned} \tag{A.18}$$

where $|T(r,\theta)| < Cr$ for sufficiently small $r > 0$ and for all θ with $C < \infty$. Using the inequalities

$$|(d-2)uJ| \leq \sqrt{\frac{|d-2|}{8}}(2(d-2)u^2 + J^2),$$

and

$$|uJ| \leq \frac{u^2}{2} + \frac{J^2}{2},$$

It is clear that for $2 < d < 10$ we can find a $\omega > 0$, a ξ_u with $-\infty < \xi_u < \xi_1$ and a $r_1 > 0$ such that for $r \leq r_1$ and $\xi \leq \xi_u$, $\theta_\xi < -\omega$.

Without loss of generality, we can take $r_1 \leq \max(r_0, 1/2)$. Choose

$$\delta = r_1 \exp[-3A(\xi_1 - \xi_u + \pi(N+1)/\omega)/2].$$

Choose a $\mathbf{y}$ such that $0 < r(\xi_1) < \delta$. Then, it is clear that $0 < r(\xi) \leq r_1$ for $\xi \in [\xi_u - \pi(N+1)/\omega, \xi_1]$ so that $\xi_2(\mathbf{y}) < \xi_u - \pi(N+1)/\omega$. Therefore, $\theta_\xi < -\omega$ and this implies that there are at least N intersections with $u = 0$, or equivalently with the line $H = 1$ for $\xi \in (\xi_2(\mathbf{y}), \xi_1]$. □

To prove the theorem, we need the following observations which can be verified by direct computation.

Observation 1 $H \equiv 0, J \equiv 0$ is a solution of (A.2) for $\xi \in (-\infty, \infty)$. Then, $(H(\xi), G(\xi)) \in \mathcal{S}$ and $\lim_{\xi\to-\infty}(H(\xi), G(\xi)) = (0,0)$. This solution has no intersections with the line $H = 1$.

Observation 2 Another exact solution for $\xi \in (-\infty, \infty)$ is given by

$$H(\xi) = \frac{2e^{2\xi}}{2(d-2) + e^{2\xi}}.$$

Then, $(H(\xi), G(\xi)) \in \mathcal{S}$ and $\lim_{\xi\to-\infty}(H(\xi), G(\xi)) = (0,0)$, and this solution has one intersection with $H = 1$.

We now have all the results needed to prove the theorem:

Proof. For $U \in [-1,1]$ Lemma A.1 gives the existence of a solution to (A.2) with $\lim_{\xi\to\infty}(H,J) = (1+U,0)$ for $\xi \in [\xi_0,\infty)$ Further, if $U \neq 0$, the solution has no intersections with the line $H = 1$in the interval $[\xi_0,\infty)$. Lemma A.4 guarantees that we can extend the solution to a set $(\xi_2(U),\infty)$ such that $(H,G) \in \mathcal{S}$ on this set. Further, $\lim_{\xi\to\xi_2(U)} H(\xi) = 0$ and $\lim_{\xi\to\xi_2(U)} G(\xi) = G_2(U) \geq 0$. We will denote the solution over the entire interval where it is in the set $\mathcal{S}$ by $\mathbf{x}(\xi,U)$ so that the dependence of the solution on U becomes explicit.

Lemma A.2, Lemma A.3 and Lemma A.5 together imply that if $G_2(U_a) > 0$, we can find an open set $\mathcal{O}$ containing U_a such that the function $N(U) \equiv$ number of intersections with the line $H = 1$ is a constant on $\mathcal{O}$. This implies that if there exists a U_b such that the function $N(U)$ is non-constant on every open set containing U_b, then $G_2(U_b) = 0$ so that $\xi_2(U_b) = \infty$.

Observations 1 and 2 give explicit solutions with $N(U_c) = 0$ and $N(U_d) = 1$ with $G_2(U_c) = G_2(U_d) = 0$. (A.9) along with $|u(\xi_0)| \leq |U_\infty|$ implies that we can choose U such that $|u(\xi_0)| + |J(\xi_0)| < \delta$ for any given $\delta > 0$. This along with Lemma A.7 gives the existence of a $U_e \in [-1,0)$ and a $U_f \in (0,1]$ with $N(U_e) > N$ and $N(U_f) > N$ for any given $N < \infty$. The corollary to Lemma A.4 implies that $N(U) < \infty$ for $U \in [-1,0)\cup(0,1]$ so that $N(U_e)$ and $N(U_f)$ are both finite. Because $N(U)$ can change only at points U_b with $G_2(U_b) = 0$, it follows that there exists $U_b \in [U_c,U_e]$ (for example) with $G_2(U_b) = 0$.

Assume that U_b is such that $G_2(U_b) = 0$ so that $\lim_{\xi\to-\infty} \mathbf{x}(\xi,U_b) = (0,0)$. Then, there exists a $\xi_n > -\infty$ such that $\mathbf{x}(\xi_n,U_b) \in \mathcal{W}$ for $\xi \in (-\infty,\xi_n+1)$ and without loss of generality, $\xi_n \leq \xi_m$ where ξ_m and $\mathcal{W}$ are as defined in the proof of Lemma A.6. Using Lemma A.2 along with the continuous dependence of the solutions $\mathbf{x}(\xi)$ of (A.3) on $\mathbf{x}(\xi_1)$ on the set $[\xi_f,\xi_i]$ for any given $\xi_f > -\infty$ and $\xi_i < \infty$, it is clear that we can find a sufficiently small neighborhood $\mathcal{V}$ of U_b such that for all $U \in \mathcal{V}$, we have $\mathbf{x}(\xi_n,U) \in \mathcal{W}$ From Lemma A.6 it follows that $N(U) = N(U_b)$ or $N(U) = N(U_b)+2$ for all $U \in \mathcal{V}$. Therefore, at points U_b where the function $N(U)$ is discontinuous, it can only change by 2.

If $U > 0$, since $H(\xi_2(U)) = 0$, we have the $N(U)$ is odd and similarly for $U < 0$, $N(U)$ is even. We have a U_c with $N(U_c) = 0$. Also, we can choose a $U_e < 0$ such that $N(U_e)$ is even, finite and larger than any given N. This along with the above arguments show that there exist $U_{2i} \in [-1,0)$ such that $G_2(U_{2i}) = 0$ for $i = 0,1,\ldots,[N/2]$. We can similarly argue for the case where $N(U_b)$ is odd and $G_2(U_b) = 0$. Because N can be arbitrarily large, we have that given $i < \infty$, there exists a $U_i \in [-1,1]$ with $N(U_i) = i$ and $G_2(U_i) = 0$ which implies that $\xi_2(U_i) = -\infty$.

Doing a linear analysis near the point $H = 0, G = 0$, we can show that if $(H,G) \to (0,0)$ as $\xi \to -\infty$, $H(\xi)e^{-2\xi} \to A$ for some finite positive constant A. Writing $\xi = \log(\eta)$ and defining the solution at $\eta = 0$ as $\lim_{\xi\to-\infty} H(\xi) = 0$ by Lemma A.3, we see that we have a countable number of solutions to (66) satisfying the boundary conditions $H(\eta)/\eta^2$ bounded as $\eta \to 0$ and $\eta\partial_\eta H = H_\xi = J \to 0$ as $\eta \to \infty$. Also, the solutions satisfy the physically motivated requirements $(H(\eta),G(\eta)) \in \mathcal{S}$, so that the

similarity solutions have a non-negative mass and density. □

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