Nekovar duality in p-adic Lie extensions

Romyar Sharifi

University of Arizona

February 22, 2011

Topological modules

Λ pro-p ring, an inverse limit of finite quotients of p-power order
Λ° its opposite ring, G profinite group

$\mathrm{Top}_{\Lambda,G}$ category of Hausdorff topological Λ-modules with a continuous commuting G-action, with continuous $\Lambda[G]$-module homomorphisms

$\mathcal{C}_{\Lambda,G}$ and $\mathcal{D}_{\Lambda,G}$ full subcategories of compact and discrete modules

Theorem

The Pontryagin dual $T^\vee = \mathrm{Hom}_{\mathrm{cts}}(T, \mathbf{Q}_p/\mathbf{Z}_p)$ *provides contravariant equivalences of categories*

$$\mathcal{C}_{\Lambda,G} \overset{\vee}{\longleftrightarrow} \mathcal{D}_{\Lambda^\circ,G},$$

with $T \cong (T^\vee)^\vee$.

For $f \in T^\vee$, $t \in T$, $\sigma \in G$, and $\lambda \in \Lambda$, we have

$$(\sigma \cdot f)(t) = f(\sigma^{-1} t) \quad \text{and} \quad (f \cdot \lambda)(t) = f(\lambda t).$$

Continuous cochains and cohomology

Proposition

a. *Every finite module in* $\mathrm{Top}_{\Lambda,G}$ *has the discrete topology.*
b. *Every object in* $\mathcal{D}_{\Lambda,G}$ *is a union of finite* $\Lambda[G]$*-submodules.*
c. *Every object in* $\mathcal{C}_{\Lambda,G}$ *is an inverse limit of finite* $\Lambda[G]$*-modules.*

For $T \in \mathcal{C}_{\Lambda,G}$ or $\mathcal{D}_{\Lambda,G}$, the complex of continuous cochains $C(G,T)$ is a complex of Λ-modules.

The cohomology groups $H^i(G,T)$ of $C(G,T)$ form a δ-functor. If

$$0 \to A \to B \to C \to 0$$

is an exact sequence in $\mathcal{C}_{\Lambda,G}$ or $\mathcal{D}_{\Lambda,G}$, then it has a continuous set-theoretic splitting, and the sequence of Λ-modules

$$H^i(G,A) \to H^i(G,B) \to H^i(G,C) \xrightarrow{\delta^i} H^{i+1}(G,A)$$

is exact, where δ^i is the usual connecting homomorphism.

Tate duality

F global field of characteristic not equal to p
G_v absolute Galois group of completion F_v for v place of F
$T \in \mathcal{C}_{\Lambda,G_v}$ or $\mathcal{D}_{\Lambda,G_v}$

Cup product pairing (with Tate cohomology if v nonarchimedean):

$$H^i(G_v, T) \times H^{2-i}(G_v, T^\vee(1)) \xrightarrow{\cup} H^2(G_v, \mathbf{Q}_p/\mathbf{Z}_p(1)) \xrightarrow{\sim} \mathbf{Q}_p/\mathbf{Z}_p.$$

If v nonarchimedean, then $H^i(G_v, T) = 0$ for all $i > 2$.

Tate duality

F global field of characteristic not equal to p
G_v absolute Galois group of completion F_v for v place of F
$T \in \mathcal{C}_{\Lambda,G_v}$ or $\mathcal{D}_{\Lambda,G_v}$

Cup product pairing (with Tate cohomology if v nonarchimedean):

$$H^i(G_v, T) \times H^{2-i}(G_v, T^\vee(1)) \xrightarrow{\cup} H^2(G_v, \mathbf{Q}_p/\mathbf{Z}_p(1)) \xrightarrow{\sim} \mathbf{Q}_p/\mathbf{Z}_p.$$

If v nonarchimedean, then $H^i(G_v, T) = 0$ for all $i > 2$.

Theorem (Tate duality)

For $i \in \mathbf{Z}$ and $v \in S$, above pairing is nondegenerate, inducing

$$H^i(G_v, T) \cong H^{2-i}(G_v, T^\vee(1))^\vee$$

as Λ-modules.

Poitou-Tate duality

S finite set of primes of F including primes over p and real places
$G_{F,S}$ Galois group of maximal unramified outside S extension of F
$T \in \mathcal{C}_{\Lambda, G_{F,S}}$ or $\mathcal{D}_{\Lambda, G_{F,S}}$

Definition

For $i \in \mathbf{Z}$, we define ith *Tate-Shafarevich group* of T by

$$Ш^i(G_{F,S}, T) = \ker \left(H^i(G_{F,S}, T) \to \bigoplus_{v \in S} H^i(G_v, T) \right).$$

Theorem (Poitou-Tate duality)

There are natural isomorphisms

$$Ш^i(G_{F,S}, T) \cong Ш^{3-i}(G_{F,S}, T^\vee(1))^\vee$$

of Λ-modules.

Poitou-Tate sequence

Theorem (Poitou-Tate)

For each $T \in \mathcal{C}_{\Lambda, G_{F,S}}$, we have an exact sequence

$$0 \longrightarrow H^0(G_{F,S}, T) \longrightarrow \bigoplus_{v \in S} H^0(G_v, T) \longrightarrow H^2(G_{F,S}, T^\vee(1))^\vee$$
$$\longrightarrow H^1(G_{F,S}, T) \longrightarrow \bigoplus_{v \in S} H^1(G_v, T) \longrightarrow H^1(G_{F,S}, T^\vee(1))^\vee$$
$$\longrightarrow H^2(G_{F,S}, T) \longrightarrow \bigoplus_{v \in S} H^2(G_v, T) \longrightarrow H^0(G_{F,S}, T^\vee(1))^\vee \longrightarrow 0.$$

and, for $i \geq 3$, isomorphisms

$$H^i(G_{F,S}, T) \xrightarrow{\sim} \bigoplus_{v \in S} H^i(G_v, T)$$

of Λ-modules.

Cones and shifts

$f\colon A \to B$ map of complexes of Λ-modules

Definition

For $i \in \mathbf{Z}$, the ith *shift* of A is the complex $A[i]$ with

$$A[i]^j = A^{i+j}, \qquad d^j_{A[i]} = (-1)^i d^{i+j}_A.$$

Definition

The *cone* of f is the complex $\mathrm{Cone}(f)$ with

$$\mathrm{Cone}(f)^i = A^{i+1} \oplus B^i, \qquad d^i_{\mathrm{Cone}(f)}(a,b) = (d^i_A(a), f(a) - d^i_B(b)).$$

The cone of f fits into an exact sequence of complexes

$$0 \to B \to \mathrm{Cone}(f) \to A[1] \to 0$$

giving rise to a long exact sequence

$$H^i(B) \to H^i(\mathrm{Cone}(f)) \to H^{i+1}(A) \xrightarrow{f^{i+1}} H^{i+1}(B).$$

Cohomology with compact support

Definition

For $T \in \mathcal{C}_{\Lambda,G_{F,S}}$ or $\mathcal{D}_{\Lambda,G_{F,S}}$, its *compactly-supported cochain complex* is

$$C_c(G_{F,S},T) = \mathrm{Cone}\Big(C(G_{F,S},T) \xrightarrow{\sum \mathrm{Res}_v} \bigoplus_{v \in S} C(G_v,T)\Big)[-1],$$

where we use Tate cochain complexes for archimedean v.

By definition, there is a long exact sequence

$$H^i_c(G_{F,S},T) \to H^i(G_{F,S},T) \to \bigoplus_{v \in S} H^i(G_v,T) \to H^{i+1}_c(G_{F,S},T)$$

$H^3_c(G_{F,S},\mathbf{Q}_p/\mathbf{Z}_p(1)) \xrightarrow{\sim} \mathbf{Q}_p/\mathbf{Z}_p$ and $H^i_c(G_{F,S},T) = 0$ for $i > 3$.

Cohomology with compact support

Definition

For $T \in \mathcal{C}_{\Lambda,G_{F,S}}$ or $\mathcal{D}_{\Lambda,G_{F,S}}$, its *compactly-supported cochain complex* is

$$C_c(G_{F,S},T) = \mathrm{Cone}\Big(C(G_{F,S},T) \xrightarrow{\sum \mathrm{Res}_v} \bigoplus_{v\in S} C(G_v,T)\Big)[-1],$$

where we use Tate cochain complexes for archimedean v.

By definition, there is a long exact sequence

$$H^i_c(G_{F,S},T) \to H^i(G_{F,S},T) \to \bigoplus_{v\in S} H^i(G_v,T) \to H^{i+1}_c(G_{F,S},T)$$

$H^3_c(G_{F,S},\mathbf{Q}_p/\mathbf{Z}_p(1)) \xrightarrow{\sim} \mathbf{Q}_p/\mathbf{Z}_p$ and $H^i_c(G_{F,S},T) = 0$ for $i > 3$.

Theorem (Poitou-Tate duality reformulated, Niziol)

Let $T \in \mathcal{C}_{\Lambda,G_{F,S}}$ or $\mathcal{D}_{\Lambda,G_{F,S}}$. For any $i \in \mathbf{Z}$, we have

$$H^i(G_{F,S},T) \cong H^{3-i}_c(G_{F,S},T^\vee(1))^\vee.$$

Derived categories

$\mathcal{C}$ abelian category
$\mathrm{Ch}(\mathcal{C})$ abelian category of chain complexes in $\mathcal{C}$

Definition

A *quasi-isomorphism* is a map $f\colon A \to B$ in $\mathrm{Ch}(\mathcal{C})$ that induces isomorphisms on cohomology $H^i(A) \xrightarrow{\sim} H^i(B)$ for all $i \in \mathbf{Z}$.

Derived categories

$\mathcal{C}$ abelian category
$\mathrm{Ch}(\mathcal{C})$ abelian category of chain complexes in $\mathcal{C}$

Definition

A *quasi-isomorphism* is a map $f\colon A \to B$ in $\mathrm{Ch}(\mathcal{C})$ that induces isomorphisms on cohomology $H^i(A) \xrightarrow{\sim} H^i(B)$ for all $i \in \mathbf{Z}$.

Definition

The *derived category* $\mathbf{D}(\mathcal{C})$ of $\mathcal{C}$ is the localized category obtained from $\mathrm{Ch}(\mathcal{C})$ by formally inverting quasi-isomorphisms, when it exists.

Remarks

1. Objects of $\mathbf{D}(\mathcal{C})$ are chain complexes, but morphisms are compositions of maps of chain complexes with inverses of quasi-isomorphisms.
2. A map $f\colon A \to B$ induces an isomorphism in $\mathbf{D}(\mathcal{C})$ if and only if every $H^i(A) \to H^i(B)$ is an isomorphism.

Bounded derived categories

We can speak of chain complexes in $\mathcal{C}$ that are bounded, bounded above, or bounded below, and the corresponding categories are denoted $\mathrm{Ch}^b(\mathcal{C})$, $\mathrm{Ch}^-(\mathcal{C})$, $\mathrm{Ch}^+(\mathcal{C})$, respectively.

Definition

The *bounded derived categories* $\mathbf{D}^b(\mathcal{C})$, $\mathbf{D}^-(\mathcal{C})$, and $\mathbf{D}^+(\mathcal{C})$ are the derived categories of $\mathrm{Ch}^b(\mathcal{C})$, $\mathrm{Ch}^-(\mathcal{C})$, and $\mathrm{Ch}^+(\mathcal{C})$, respectively.

Theorem

a. *If $\mathcal{C}$ has enough injectives, then $\mathbf{D}^+(\mathcal{C})$ exists and every object in it is isomorphic to a bounded below complex of injectives.*
b. *If $\mathcal{C}$ has enough projectives, then $\mathbf{D}^-(\mathcal{C})$ exists and every object in it is isomorphic to a bounded above complex of projectives.*

Remark

We often view derived objects as defined up to isomorphism in the derived category, in particular allowing us to replace our input complexes with quasi-isomorphic complexes.

Derived cochain complexes

Let T be a bounded complex of objects in $\mathcal{C}_{\Lambda,G_{F,S}}$ or $\mathcal{D}_{\Lambda,G_{F,S}}$.

Remark

The complex $C(G_{F,S},T)$ of continuous $G_{F,S}$-cochains of T is the total complex of a bicomplex with terms $C^i(G_{F,S},T^j)$.

Definition

For $T \in \mathrm{Ch}^b(\mathcal{C}_{\Lambda,G_{F,S}})$ or $\mathrm{Ch}^b(\mathcal{D}_{\Lambda,G_{F,S}})$,

$$\mathbf{R}\Gamma(G_{F,S},T), \quad \mathbf{R}\Gamma_c(G_{F,S},T), \text{ and } \mathbf{R}\Gamma(G_v,T)$$

denote the complexes $C(G_{F,S},T)$, $C_c(G_{F,S},T)$, and $C(G_v,T)$, respectively, considered as objects of $\mathbf{D}(\mathrm{Mod}_\Lambda)$.

Remark

If F has no real places or p is odd, then all of the latter derived objects may be viewed as objects in $\mathbf{D}^b(\mathrm{Mod}_\Lambda)$.

Derived homomorphism groups

Let Λ, Ω, Σ, and Ξ be pro-p algebras over a commutative pro-p ring R. We use $\mathrm{Mod}_{\Lambda-\Omega}$ to denote the category of $\Lambda \otimes_R \Omega^\circ$-modules.

Definition

Let $A \in \mathrm{Ch}(\mathrm{Mod}_{\Lambda-\Omega})$ and $B \in \mathrm{Ch}(\mathrm{Mod}_{\Lambda-\Sigma})$.

1. Suppose that Ω is R-projective and A is bounded above. The *derived homomorphism complex*

$$\mathbf{R}\mathrm{Hom}_\Lambda(A, B) \in \mathbf{D}(\mathrm{Mod}_{\Omega-\Sigma}).$$

   is the total complex $\mathrm{Hom}_\Lambda(P, B)$ for a complex of Λ-projective $\Lambda \otimes_R \Omega^\circ$-modules P with a quasi-isomorphism $P \xrightarrow{\sim} A$.
2. Suppose that Σ is R-flat and B is bounded below. We similarly define $\mathbf{R}\mathrm{Hom}_\Lambda(A, B)$ to be $\mathrm{Hom}_\Lambda(A, I)$ for a complex of Λ-injective $\Lambda \otimes_R \Sigma^\circ$-modules $I \xleftarrow{\sim} B$.

The cohomology groups of $\mathbf{R}\mathrm{Hom}_\Lambda(A, B)$ are the (hyper-)Ext functors $\mathrm{Ext}^i_\Lambda(A, B)$.

Exact triangles

Let $f\colon A \to B$ in $\mathrm{Ch}(\mathcal{C})$. Let $C = \mathrm{Cone}(f)$. Recall that we have an exact sequence

$$0 \to B \xrightarrow{\iota} C \xrightarrow{\pi} A[1] \to 0$$

with connecting homomorphism determined by f.

The maps of complexes

$$A \xrightarrow{f} B \xrightarrow{\iota} C \xrightarrow{\pi} A[1], \tag{1}$$

are together known as an *exact triangle*.

Remark

More generally, there is a notion of isomorphism of exact triangles, which allows us to consider like sequences of maps as in (1) that are *isomorphic* to those constructed directly from cones.

Remark

An exact triangle is not an exact sequence of complexes.

Poitou-Tate duality in derived categories

Theorem (Poitou-Tate sequence, as stated by Lim)

For $T \in \mathbf{D}^b(\mathcal{C}_{\Lambda, G_{F,S}})$, there is an isomorphism of exact triangles in $\mathbf{D}(\mathrm{Mod}_\Lambda)$

$$\begin{array}{ccc}
\mathbf{R}\Gamma_c(G_{F,S}, T) & \xrightarrow{\sim} & \mathbf{R}\mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{R}\Gamma(G_{F,S}, T^\vee(1)), \mathbf{Q}_p/\mathbf{Z}_p)[-3] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma(G_{F,S}, T) & \xrightarrow{\sim} & \mathbf{R}\mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{R}\Gamma_c(G_{F,S}, T^\vee(1)), \mathbf{Q}_p/\mathbf{Z}_p)[-3] \\
\downarrow & & \downarrow \\
\bigoplus_{v \in S} \mathbf{R}\Gamma(G_v, T) & \xrightarrow{\sim} & \bigoplus_{v \in S} \mathbf{R}\mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{R}\Gamma(G_v, T^\vee(1)), \mathbf{Q}_p/\mathbf{Z}_p)[-2] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma_c(G_{F,S}, T)[1] & \xrightarrow{\sim} & \mathbf{R}\mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{R}\Gamma(G_{F,S}, T^\vee(1)), \mathbf{Q}_p/\mathbf{Z}_p)[-2]
\end{array}$$

Remark

$\mathbf{R}\mathrm{Hom}_{\mathbf{Z}_p}(\,\cdot\,, \mathbf{Q}_p/\mathbf{Z}_p)$ is simply $\mathrm{Hom}_{\mathbf{Z}_p}(\,\cdot\,, \mathbf{Q}_p/\mathbf{Z}_p)$ since the latter functor is exact, as $\mathbf{Q}_p/\mathbf{Z}_p$ is $\mathbf{Z}_p$-injective.

Duality over $\mathbf{Z}_p$

What if we want a duality between finitely generated Λ-modules?

Suppose $\Lambda = \mathbf{Z}_p$. Well, $\mathrm{Hom}_{\mathbf{Z}_p}(T, \mathbf{Z}_p) = 0$ if T is finite.
So $\mathrm{Hom}_{\mathbf{Z}_p}(\,\cdot\,, \mathbf{Z}_p)$ does not provide a good duality functor.
However, $\mathbf{R}\mathrm{Hom}_{\mathbf{Z}_p}(T, \mathbf{Z}_p)$ does!

Duality over $\mathbf{Z}_p$

What if we want a duality between finitely generated Λ-modules?

Suppose $\Lambda = \mathbf{Z}_p$. Well, $\mathrm{Hom}_{\mathbf{Z}_p}(T, \mathbf{Z}_p) = 0$ if T is finite.
So $\mathrm{Hom}_{\mathbf{Z}_p}(\,\cdot\,, \mathbf{Z}_p)$ does not provide a good duality functor.
However, $\mathbf{R}\mathrm{Hom}_{\mathbf{Z}_p}(T, \mathbf{Z}_p)$ does!

Idea

Replace $\mathbf{Z}_p$ by a quasi-isomorphic complex J of injective $\mathbf{Z}_p$-modules:

$$\mathbf{Z}_p \xrightarrow{\sim} J = [\mathbf{Q}_p \to \mathbf{Q}_p/\mathbf{Z}_p].$$

So $\mathbf{R}\mathrm{Hom}_{\mathbf{Z}_p}(T, \mathbf{Z}_p) = [\mathrm{Hom}_{\mathbf{Z}_p}(T, \mathbf{Q}_p) \to \mathrm{Hom}_{\mathbf{Z}_p}(T, \mathbf{Q}_p/\mathbf{Z}_p)]$.

E.g., if T is finite, then we obtain the complex

$$\mathbf{R}\mathrm{Hom}_{\mathbf{Z}_p}(T, \mathbf{Z}_p) = T^\vee[-1]$$

concentrated in degree 1, while if $T = \mathbf{Z}_p$, then

$$\mathbf{R}\mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{Z}_p, \mathbf{Z}_p) = [\mathbf{Q}_p \to \mathbf{Q}_p/\mathbf{Z}_p] \simeq \mathbf{Z}_p.$$

Modified Čech complex

Let R be a complete commutative local noetherian ring with finite residue field of characteristic p, maximal ideal $\mathfrak{m}$, and Krull dim d. Let $x_1, \ldots, x_d \in \mathfrak{m}$ be such that $R/(x_1, \ldots, x_d)$ has Krull dim 0. For $x \in R$, let $R_x = R[x^{-1}]$.

Definition

The *modified Čech complex* for R is the complex

$$C_R = \left[R \to \bigoplus_i R_{x_i} \to \bigoplus_{i<j} R_{x_i x_j} \to \cdots \to R_{x_1 \ldots x_d}\right][d]$$

with appropriate signs (i.e., $(-1)^{s+1} \colon R_{x_{i_1} \cdots \widehat{x_{i_s}} \cdots x_{i_t}} \to R_{x_{i_1} \cdots x_{i_t}}$).

Remark

The complex $J_R = C_R^\vee$ consists of of R-injectives, and it is quasi-isomorphic to a bounded complex of finitely generated R-modules.

Grothendieck duality

Example

$R = \mathbf{Z}_p$, $x_1 = p$. Then

$$C_{\mathbf{Z}_p} = [\mathbf{Z}_p \to \mathbf{Q}_p][1] \quad \text{and} \quad J_{\mathbf{Z}_p} = [\mathbf{Q}_p \to \mathbf{Q}_p/\mathbf{Z}_p] \simeq \mathbf{Z}_p.$$

Grothendieck duality

Example

$R = \mathbf{Z}_p$, $x_1 = p$. Then

$$C_{\mathbf{Z}_p} = [\mathbf{Z}_p \to \mathbf{Q}_p][1] \quad \text{and} \quad J_{\mathbf{Z}_p} = [\mathbf{Q}_p \to \mathbf{Q}_p/\mathbf{Z}_p] \simeq \mathbf{Z}_p.$$

Definition

The *dualizing complex* $\omega_R \in \mathbf{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R)$ is the object represented by J_R in the derived category of complexes that are quasi-isomorphic to bounded complexes of finitely generated R-modules.

Theorem (Grothendieck duality)

For $T \in \mathbf{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R)$, there exists a canonical quasi-isomorphism

$$T \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(T, \omega_R), \omega_R).$$

Remark

If R is regular (or Gorenstein), then $\omega_R \simeq R$.

Nekovar duality

Let $\mathcal{A}_{R,G_{F,S}}$ denote the full subcategory of $\mathcal{C}_{R,G_{F,S}}$ of modules that are finitely generated over R, which is equivalent to the full subcategory of abstract $R[G_{F,S}]$-modules with the same objects.

Let $T \in \mathrm{Ch}^b(\mathcal{A}_{R,G_{F,S}})$, and choose a subcomplex $T^* \in \mathrm{Ch}^b(\mathcal{A}_{R,G_{F,S}})$ of $\mathrm{Hom}_R(T, J_R)$ that is quasi-isomorphic to $\mathrm{Hom}_R(T, J_R)$.

Nekovar duality

Let $\mathcal{A}_{R,G_{F,S}}$ denote the full subcategory of $\mathcal{C}_{R,G_{F,S}}$ of modules that are finitely generated over R, which is equivalent to the full subcategory of abstract $R[G_{F,S}]$-modules with the same objects.

Let $T \in \mathrm{Ch}^b(\mathcal{A}_{R,G_{F,S}})$, and choose a subcomplex $T^* \in \mathrm{Ch}^b(\mathcal{A}_{R,G_{F,S}})$ of $\mathrm{Hom}_R(T, J_R)$ that is quasi-isomorphic to $\mathrm{Hom}_R(T, J_R)$.

The map $T \otimes_R T^*(1) \to J_R(1)$ on the total tensor product allows us to construct a cup product. We then take the adjoint map to obtain a morphism as in the following theorem.

Theorem (Nekovar)

There is a natural isomorphism in $\mathbf{D}(\mathrm{Mod}_\Lambda)$ *given by*

$$\mathbf{R}\Gamma(G_{F,S}, T) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_c(G_{F,S}, T^*(1)), \omega_R)[-3].$$

Remark

The theorem yields the (hypercohomology) spectral sequence

$$\mathrm{Ext}^i_R(H^j_c(G_{F,S}, T^*(1)), \omega_R) \Rightarrow H^{3-i-j}(G_{F,S}, T).$$

Induced modules

Let F_∞/F be a p-adic Lie extension that is unramified outside S.
Set $\Gamma = \mathrm{Gal}(F_\infty/F)$ and $\Lambda = R[[\Gamma]]$.
Λ is noetherian and pro-p, as well as R-flat.

View Λ as a $\Lambda^\circ[G_{F,S}]$-module with Γ acting by right multiplication and $\sigma \in G_{F,S}$ by left multiplication by the image of σ in Γ.

For any complex of $\Lambda[G_{F,S}]$-modules (or $\Lambda^\circ[G_{F,S}]$-modules) M, we let M^ι denote the complex of $\Lambda^\circ[G_{F,S}]$-modules (or $\Lambda[G_{F,S}]$-modules) with the same $G_{F,S}$-actions but with $\gamma \in \Gamma$ acting by γ^{-1}.

Definition

For $T \in \mathrm{Ch}^b(\mathcal{C}_{R,G_{F,S}})$, we set

$$\mathcal{F}_\Gamma(T) = \Lambda^\iota \hat{\otimes}_R T \in \mathrm{Ch}^b(\mathcal{C}_{\Lambda,G_{F,S}}),$$

and for $A \in \mathrm{Ch}^b(\mathcal{D}_{R,G_{F,S}})$, we set

$$F_\Gamma(A) = \mathrm{Hom}_{R,\mathrm{cts}}(\Lambda, A) \in \mathrm{Ch}^b(\mathcal{D}_{\Lambda,G_{F,S}}).$$

Remark

Shapiro's lemma provides isomorphisms

$$H^i(G_{F,S}, \mathcal{F}_\Gamma(T)) \cong \varprojlim_{E \subset F_\infty} H^i(G_{E,S}, T),$$
$$H^i(G_{F,S}, F_\Gamma(A)) \cong \varinjlim_{E \subset F_\infty} H^i(G_{E,S}, A)$$

(and also for local and compactly-supported cohomology).

Theorem (Lim)

For $T \in \mathrm{Ch}^b(\mathcal{C}_{R,G_{F,S}})$, we have natural isomorphisms in $\mathbf{D}^b(\mathrm{Mod}_\Lambda)$ given by

$$\mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_\Gamma(T)) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{R}\Gamma_c(G_{F,S}, F_\Gamma(T^\vee)^\iota(1)), \mathbf{Q}_p/\mathbf{Z}_p)[-3],$$

and similarly for $\mathbf{R}\Gamma_c(G_{F,S}, \mathcal{F}_\Gamma(T))$ and $\mathbf{R}\Gamma(G_v, \mathcal{F}_\Gamma(T))$ for $v \in S$.

Duality between induced modules

From now on, T is an object of $\mathrm{Ch}^b(\mathcal{A}_{R,G_{F,S}})$.

Proposition

We have a natural isomorphism

$$\mathcal{F}_\Gamma(T) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathcal{F}_\Gamma(T^*)^\iota, \Lambda \otimes_R \omega_R).$$

in $\mathbf{D}^b(\mathrm{Mod}_\Lambda)$.

Remark

The above can be seen as an isomorphism in the bounded derived category of the ind-category $\mathcal{I}_{\Lambda,G_{F,S}}$ of $\mathcal{A}_{\Lambda,G_{F,S}}$. There, we have a derived bifunctor

$$\mathbf{R}\mathrm{Hom}_{\Lambda^\circ,\mathrm{cts}}(-,-)\colon \mathbf{D}^-(\mathcal{C}_{R,G_{F,S}}) \times \mathbf{D}^+(\mathcal{I}_{\Lambda-\Lambda}) \to \mathbf{D}^+(\mathcal{I}_{\Lambda,G_{F,S}}).$$

Nekovar duality in p-adic Lie extensions

Would like a Nekovar-type duality up the Iwasawa tower between cohomology groups with coefficients in $\mathcal{F}_\Gamma(T)$ and $\mathcal{F}_\Gamma(T^*)^\iota(1)$.

Nekovar duality in p-adic Lie extensions

Would like a Nekovar-type duality up the Iwasawa tower between cohomology groups with coefficients in $\mathcal{F}_\Gamma(T)$ and $\mathcal{F}_\Gamma(T^*)^\iota(1)$.

Problem

The R-algebra Λ can be noncommutative, and we do not know if it has a nice enough dualizing complex of bimodules for our application.

Nekovar duality in p-adic Lie extensions

Would like a Nekovar-type duality up the Iwasawa tower between cohomology groups with coefficients in $\mathcal{F}_\Gamma(T)$ and $\mathcal{F}_\Gamma(T^*)^\iota(1)$.

Problem

The R-algebra Λ can be noncommutative, and we do not know if it has a nice enough dualizing complex of bimodules for our application.

Idea

$\Lambda \otimes_R \omega_R$ plays the role of ω_R for cohomology of induced modules.

Nekovar duality in p-adic Lie extensions

Would like a Nekovar-type duality up the Iwasawa tower between cohomology groups with coefficients in $\mathcal{F}_\Gamma(T)$ and $\mathcal{F}_\Gamma(T^*)^\iota(1)$.

Problem

The R-algebra Λ can be noncommutative, and we do not know if it has a nice enough dualizing complex of bimodules for our application.

Idea

$\Lambda \otimes_R \omega_R$ plays the role of ω_R for cohomology of induced modules.

Remark

Fukaya and Kato proved a result of this form for a more general class of (adic rings) Λ: e.g.,

$$\mathbf{R}\Gamma_c(G_{F,S}, X) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma(G_{F,S}, X^*(1)), \Lambda)[-3]$$

for any complex X of finitely generated projective Λ-modules with continuous commuting actions of $G_{F,S}$, and its Λ-dual X^*. In our setting, this applies to $X = \mathcal{F}_\Gamma(T)$ when T consists of R-projectives.

Main result

Theorem (Lim-S.)

We have a natural isomorphism of exact triangles in $\mathbf{D}(\mathrm{Mod}_\Lambda)$ *given by*

$$\begin{array}{ccc}
\mathbf{R}\Gamma_c(G_{F,S}, \mathcal{F}_\Gamma(T)) & \xrightarrow{\sim} & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R \omega_R)[-3] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_\Gamma(T)) & \xrightarrow{\sim} & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_{F,S}, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R \omega_R)[-3] \\
\downarrow & & \downarrow \\
\bigoplus_{v\in S} \mathbf{R}\Gamma(G_v, \mathcal{F}_\Gamma(T)) & \xrightarrow{\sim} & \bigoplus_{v\in S} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma(G_v, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R \omega_R)[-2] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma_c(G_{F,S}, \mathcal{F}_\Gamma(T))[1] & \xrightarrow{\sim} & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R \omega_R)[-2]
\end{array}$$

Remark

Any two of the first three of these isomorphisms implies the third: we focus here on the second. The proof of the first involves additional technicalities to deal with cohomologically unbounded complexes in the case that $p = 2$ and F has real places.

Derived tensor products

Definition

Suppose that Ω is R-flat. For $A \in \mathrm{Ch}^-(\mathrm{Mod}_{\Omega-\Lambda})$ and $B \in \mathrm{Ch}(\mathrm{Mod}_{\Lambda-\Sigma})$, the *derived tensor product*

$$A \otimes^{\mathbf{L}}_{\Lambda} B \in \mathbf{D}(\mathrm{Mod}_{\Omega-\Sigma})$$

is the total tensor product $P \otimes_\Lambda B$ for a bounded above complex of Λ°-flat $\Omega \otimes_R \Lambda^\circ$-modules $P \xrightarrow{\sim} A$.

Cohomology groups of $A \otimes^{\mathbf{L}}_{\Lambda} B$ are the higher (hyper)-Tor functors.

Remark

If Ω is projective in $\mathcal{C}_R$, we also have derived completed tensor products: for $A \in \mathrm{Ch}^-(\mathcal{C}_{\Omega-\Lambda})$ and $B \in \mathrm{Ch}(\mathcal{C}_{\Lambda-\Sigma})$, we have $A \,\hat{\otimes}^{\mathbf{L}}_{R} B \in \mathbf{D}(\mathcal{C}_{\Omega-\Sigma})$. Can also incorporate $G_{F,S}$-actions.

Some derived isomorphisms

Lemma A

Let Ξ *be* R*-flat and* Σ *be* R*-projective. Let* $A \in \mathrm{Ch}^-(\mathrm{Mod}_{\Omega-\Lambda})$, $B \in \mathrm{Ch}(\mathrm{Mod}_{\Lambda-\Sigma})$, *and* $C \in \mathrm{Ch}^+(\mathrm{Mod}_{\Omega-\Xi})$. *Then we have an isomorphism in* $\mathbf{D}(\mathrm{Mod}_{\Sigma-\Xi})$ *given by*

$$\mathbf{R}\mathrm{Hom}_\Omega(A \otimes^{\mathbf{L}}_\Lambda B, C) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_\Lambda(B, \mathbf{R}\mathrm{Hom}_\Omega(A, C)).$$

Lemma B

Let Ω *be* R*-flat and* Ξ *be* R*-projective. Let* $A \in \mathrm{Ch}^b(\mathrm{Mod}_{\Omega-\Sigma})$, $B \in \mathrm{Ch}^-(\mathrm{Mod}_{\Xi-\Lambda})$, *and* $C \in \mathrm{Ch}^+(\mathrm{Mod}_{\Sigma-\Lambda})$. *Suppose that* A *is quasi-isomorphic to a bounded complex with* Σ°*-flat terms and that* B *is quasi-isomorphic to a bounded above complex with terms that are finitely presented and projective over* Λ°. *Then we have an isomorphism in* $\mathbf{D}(\mathrm{Mod}_{\Omega-\Xi})$ *given by*

$$A \otimes^{\mathbf{L}}_\Sigma \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(B, C) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(B, A \otimes^{\mathbf{L}}_\Sigma C).$$

Descent spectral sequence

Theorem (Descent spectral sequence, Lim-S.)

There is a natural isomorphism in $\mathbf{D}(\mathrm{Mod}_R)$ *given by*

$$R \otimes^{\mathbf{L}}_{\Lambda} \mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_{\Gamma}(T)) \xrightarrow{\sim} \mathbf{R}\Gamma(G_{F,S}, T).$$

Remark

Have similar descent sequences in local, compactly supported cases.

Descent spectral sequence

Theorem (Descent spectral sequence, Lim-S.)

There is a natural isomorphism in $\mathbf{D}(\mathrm{Mod}_R)$ *given by*

$$R \otimes^{\mathbf{L}}_{\Lambda} \mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_{\Gamma}(T)) \xrightarrow{\sim} \mathbf{R}\Gamma(G_{F,S}, T).$$

Remark

Have similar descent sequences in local, compactly supported cases.

Corollary

The cohomology groups $H^i(G_{F,S}, \mathcal{F}_{\Gamma}(T))$ *are finitely generated* Λ*-modules.*

Remark

The descent sequence was also proven by Fukaya-Kato in the above-mentioned setting.

Reduction to torsion-free coefficient rings

Let S be a complete commutative local noetherian ring with R as a quotient. Let $\Omega = S[[\Gamma]]$. Let $\mathfrak{m}_S$ be the maximal ideal of S, and let

$$d = \dim_k \mathfrak{m}_S/\mathfrak{m}_S^2 - \dim_k \mathfrak{m}/\mathfrak{m}^2.$$

Remarks

1. $\mathbf{R}\mathrm{Hom}_R(T, \omega_S) \cong \mathbf{R}\mathrm{Hom}_R(T, \omega_R)[-d]$
2. $\Omega^\iota \otimes_S T \cong \Lambda^\iota \otimes_R T = \mathcal{F}_\Gamma(T)$

Proposition

We have the following commutative diagram in $\mathbf{D}(\mathrm{Mod}_\Sigma)$*:*

$$\begin{array}{ccc} \mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_\Gamma(T)) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_{F,S}, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R \omega_R)[-3] \\ \downarrow \wr & & \downarrow \wr \\ \mathbf{R}\Gamma(G_{F,S}, \Omega^\iota \otimes_S T) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Omega^\circ}(\mathbf{R}\Gamma_c(G_{F,S}, \Omega \otimes_S T^*[-d](1)), \Omega \otimes_S \omega_S)[-3]. \end{array}$$

Reduction to torsion-free coefficient rings

Let S be a complete commutative local noetherian ring with R as a quotient. Let $\Omega = S[[\Gamma]]$. Let $\mathfrak{m}_S$ be the maximal ideal of S, and let

$$d = \dim_k \mathfrak{m}_S/\mathfrak{m}_S^2 - \dim_k \mathfrak{m}/\mathfrak{m}^2.$$

Remarks

1. $\mathbf{R}\mathrm{Hom}_R(T, \omega_S) \cong \mathbf{R}\mathrm{Hom}_R(T, \omega_R)[-d]$
2. $\Omega^\iota \otimes_S T \cong \Lambda^\iota \otimes_R T = \mathcal{F}_\Gamma(T)$

Proposition

We have the following commutative diagram in $\mathbf{D}(\mathrm{Mod}_\Sigma)$*:*

$$\begin{array}{ccc}
\mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_\Gamma(T)) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_{F,S}, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R \omega_R)[-3] \\
\downarrow \wr & & \downarrow \wr \\
\mathbf{R}\Gamma(G_{F,S}, \Omega^\iota \otimes_S T) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Omega^\circ}(\mathbf{R}\Gamma_c(G_{F,S}, \Omega \otimes_S T^*[-d](1)), \Omega \otimes_S \omega_S)[-3].
\end{array}$$

Remark

The ring R is a quotient of a polynomial ring in $\dim_k \mathfrak{m}/\mathfrak{m}^2$ variables over the Witt vectors of k, so we may assume that R is p-torsion free.

Idea of proof for torsion-free coefficient rings

To show:

$$\mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_\Gamma(T)) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_{F,S}, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R \omega_R)[-3].$$

Know:

$$\mathbf{R}\Gamma(G_{F,S}, T) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_c(G_{F,S}, T^*(1)), \omega_R)[-3].$$

Idea of proof for torsion-free coefficient rings

To show:

$$\mathbf{R}\Gamma(G_{F,S},\mathcal{F}_\Gamma(T)) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_{F,S},\mathcal{F}_\Gamma(T^*)^\iota(1)),\Lambda\otimes_R\omega_R)[-3].$$

Know:

$$\mathbf{R}\Gamma(G_{F,S},T) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_c(G_{F,S},T^*(1)),\omega_R)[-3].$$

Idea

Induction working modulo powers of the augmentation ideal I of Λ.

Note that $\Lambda = \varprojlim \Lambda/I^n$.
Moreover, $\Lambda/I \cong R$, and I^n/I^{n+1} is a finitely generated R-module with an R-flat resolution

$$[I^{n+1}\to I^n] \xrightarrow{\sim} I^n/I^{n+1}.$$

Finally, note that I^n/I^{n+1} (unlike Λ) has a trivial $G_{F,S}$-action.

Inductive step

For $0 \le m < n$, set $\mathcal{F}_{I^m/I^n}(T) = [I^n \to I^m] \otimes_\Lambda \mathcal{F}_\Gamma(T)$.

We have a commutative diagram of exact triangles:

$$\begin{array}{ccc}
\mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_{I^n/I^{n+1}}(T)) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_{F,S}, \mathcal{F}_\Gamma(T^*)^\iota(1)), I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R)[-3] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_{\Lambda/I^{n+1}}(T)) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_{F,S}, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R)[-3] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_{\Lambda/I^n}(T)) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_{F,S}, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda/I^n \otimes^{\mathbf{L}}_R \omega_R)[-3] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_{I^n/I^{n+1}}(T))[1] & \to & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_{F,S}, \mathcal{F}_\Gamma(T^*)^\iota(1)), I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R)[-2]
\end{array}$$

When $n = 1$, the third horizontal map becomes Nekovar's theorem after applying the descent spectral sequence.

By induction, the second horizontal map is an isomorphism if we can show that the first is (by the five lemma).

End of the proof

Using the flatness of $[I^{n+1} \to I^n]$ and the triviality of the $G_{F,S}$-action on I^n/I^{n+1}, we obtain a natural commutative diagram

$$\begin{array}{ccc}
I^n/I^{n+1} \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma(G_{F,S}, T) & \xrightarrow{\sim} & I^n/I^{n+1} \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_c(G_{F,S}, T^*(1)), \omega_R)[-3] \\
\downarrow \wr & & \text{Lemma B} \downarrow \wr \\
\mathbf{R}\Gamma(G_{F,S}, I^n/I^{n+1} \hat{\otimes}^{\mathbf{L}}_R T) & \longrightarrow & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_c(G_{F,S}, T^*(1)), I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R)[-3] \\
& & \text{Descent} \downarrow \wr \\
\wr & & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_c(G_{F,S}, \mathcal{F}_\Gamma(T^*)^\iota(1)) \otimes^{\mathbf{L}}_\Lambda R, I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R)[-3] \\
& & \text{Lemma A} \downarrow \wr \\
\mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_{I^n/I^{n+1}}(T)) & \to & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_{F,S}, \mathcal{F}_\Gamma(T^*)^\iota(1)), I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R)[-3].
\end{array}$$

The upper isomorphism follows from Nekovar's theorem.

End of the proof

Using the flatness of $[I^{n+1} \to I^n]$ and the triviality of the $G_{F,S}$-action on I^n/I^{n+1}, we obtain a natural commutative diagram

$$\begin{array}{ccc}
I^n/I^{n+1} \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma(G_{F,S}, T) & \xrightarrow{\sim} & I^n/I^{n+1} \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_c(G_{F,S}, T^*(1)), \omega_R)[-3] \\
\downarrow \wr & & \text{Lemma B} \downarrow \wr \\
\mathbf{R}\Gamma(G_{F,S}, I^n/I^{n+1} \hat{\otimes}^{\mathbf{L}}_R T) & \longrightarrow & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_c(G_{F,S}, T^*(1)), I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R)[-3] \\
 & & \text{Descent} \downarrow \wr \\
\downarrow \wr & & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_c(G_{F,S}, \mathcal{F}_\Gamma(T^*)^\iota(1)) \otimes^{\mathbf{L}}_\Lambda R, I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R)[-3] \\
 & & \text{Lemma A} \downarrow \wr \\
\mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_{I^n/I^{n+1}}(T)) & \to & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_{F,S}, \mathcal{F}_\Gamma(T^*)^\iota(1)), I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R)[-3].
\end{array}$$

The upper isomorphism follows from Nekovar's theorem.

To finish the proof, one then needs only to pass to the inverse limit over n of the isomorphisms

$$\mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_{\Lambda/I^n}(T)) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_{F,S}, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda/I^n \otimes^{\mathbf{L}}_R \omega_R)[-3].$$

For this, we resolve the compactly supported complex by a complex of finitely generated projective Λ°-modules, which allows us to pass the inverse limit through the homomorphism complex.

The other isomorphism

We end with a few remarks on the derivation of the isomorphism

$$\mathbf{R}\Gamma_c(G_{F,S}, \mathcal{F}_\Gamma(T)) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R \omega_R)[-3].$$

Problem

If $p = 2$ and S has real places, then $\mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_\Gamma(T^*)^\iota(1))$ and $\mathbf{R}\Gamma(G_{F,S}, T^*(1))$ will not always be cohomologically bounded above.

This causes issues in the application of Lemmas A and B, as well as in the passage to the limit, on the previous slide.

The other isomorphism

We end with a few remarks on the derivation of the isomorphism

$$\mathbf{R}\Gamma_c(G_{F,S}, \mathcal{F}_\Gamma(T)) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R \omega_R)[-3].$$

Problem

If $p = 2$ and S has real places, then $\mathbf{R}\Gamma(G_{F,S}, \mathcal{F}_\Gamma(T^*)^\iota(1))$ and $\mathbf{R}\Gamma(G_{F,S}, T^*(1))$ will not always be cohomologically bounded above.

This causes issues in the application of Lemmas A and B, as well as in the passage to the limit, on the previous slide.

Solution

Every complex is quasi-isomorphic to direct limits of complexes of bounded above projectives via maps that are term-wise split injective.

The derived homomorphism and tensor product functors can be computed in an unbounded left variable using these, allowing one to generalize Lemmas A and B as needed. The passage to the limit requires only a slightly more subtle argument.