

Assignment 2

Math 514B, Spring 2012

Due Friday, Feb. 17

1. Let K be a local field with uniformizer π and set S of representatives of its residue field. Show that every element of $U_1(K)$ may be written uniquely as a convergent product

$$\prod_{i=1}^{\infty} (1 + c_i \pi^i)$$

with each $c_i \in S$.

2. Let p be a prime. Show that the image of $\{1 - p, p\}$ generates $\mathbb{Q}_p^\times / \mathbb{Q}_p^{\times p}$ for odd p and that the image of $\{-1, 2, 3\}$ generates $\mathbb{Q}_2^\times / \mathbb{Q}_2^{\times 2}$.

3. Show that $\mathbb{Q}_p(\mu_p, a^{1/p}) / \mathbb{Q}_p(\mu_p)$ is unramified and nontrivial for an $a \in U_1(\mathbb{Q}_p(\mu_p))$ if and only if $a = 1 + u(1 - \zeta_p)^p$ for some unit $u \in \mathbb{Z}_p[\mu_p]^\times$, where ζ_p is a fixed primitive p th root of unity.

4. Let L/K be a Galois extension of local fields. Show that L may be written as a compositum of an unramified extension of K and a totally ramified extension of K and that these fields are linearly disjoint. (Hint: See Lemma 1.1.7 in the notes.)

5. Consider the field extension $L = \mathbb{Q}(\mu_{p^\infty}, \sqrt[p^\infty]{p})$ of \mathbb{Q} given by adjoining to \mathbb{Q} all p -power roots of unity and all p -power roots of p . Show that $\text{Gal}(L/\mathbb{Q})$ is isomorphic to a semi-direct product $\mathbb{Z}_p \rtimes \mathbb{Z}_p^\times$, where the conjugation action of \mathbb{Z}_p^\times on \mathbb{Z}_p in the semi-direct product is given by left multiplication inside \mathbb{Z}_p .

6. Show that $\mathbb{Q}(\sqrt[p]{-4})/\mathbb{Q}$ is an abelian extension of \mathbb{Q} of degree 2.

7. Let p be a prime and K be a field of characteristic not p . Let $a \in K^\times$ with $a \notin K^{\times p}$.

a. Show that $\sqrt[p]{a}$ has minimal polynomial $x^p - a$ over K .

b. Show that K contains μ_p if $K(\sqrt[p]{a})/K$ is Galois. (Hint: Consider the action of σ^p .)