Nekovář duality over p-adic Lie extensions of global fields

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1 Introduction

In [Nek], Nekovář gave formulations of analogues of Tate and Poitou-Tate duality for finitely generated modules over a complete commutative local Noetherian ring R with finite residue field of characteristic a fixed prime p. In the usual formulation of these dualities, one takes the Pontryagin dual, which does not in general preserve the property of finite generation. Nekovář takes the dual with respect to a dualizing complex of Grothendieck so as to have a duality between bounded complexes of R-modules with finitely generated cohomology groups. This paper is devoted to a generalization of this result to the setting of nonabelian p-adic Lie extensions.

Recall that a dualizing complex ω_R is a bounded complex of R-modules with cohomology finitely generated over R that has the property that for every complex M of finitely generated R-modules, the Grothendieck dual $\mathbf{R}\mathrm{Hom}_R(M, \omega_R)$ in the derived category of R-modules $\mathbf{D}(\mathrm{Mod}_R)$ has finitely generated cohomology, and moreover, the canonical morphism

$$M \longrightarrow \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, \omega_R), \omega_R)$$

is an isomorphism in $\mathbf{D}(\mathrm{Mod}_R)$. Such a complex exists and is unique up to quasi-isomorphism and translation (see [Har]).

One can choose ω_R to be a bounded complex of injectives, in which case the derived homomorphism complexes are represented by the complexes of homomorphisms themselves. If R is regular, then R itself, as a complex concentrated in degree 0, is a dualizing complex, but R is not in general R-injective. If $R = \mathbb{Z}_p$, for instance, then the complex $[\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p]$ concentrated in degrees -1 and 0 provides a complex of injective $\mathbb{Z}_p$-modules quasi-isomorphic to $\mathbb{Z}_p$.

Let us explain Nekovář's theorem, and our generalization of it, in the setting of Poitou-Tate duality. Let F be a global field with characteristic not equal to p. Let S be a finite set of primes of F that, if F is a number field, contains all primes over p and any real places, and let $G_{F,S}$ denote the Galois group of the maximal unramified outside S extension of p. We remark that a finitely generated R-module has a canonical topology arising from its filtration by powers of the maximal ideal of R. Let T be a bounded complex in the category of finitely generated (topological) R-modules with R-linear continuous $G_{F,S}$-actions. We use $\mathbf{R}\Gamma(G_{F,S}, T)$ to denote the object in $\mathbf{D}(\mathrm{Mod}_R)$ corresponding to the the complex of continuous $G_{F,S}$-cochains with coefficients in T, and we use $\mathbf{R}\Gamma_{(c)}(G_{F,S}, T)$ to denote the derived object attached to the complex of continuous compactly supported cochains (using Tate groups for real places in its definition as a cone: see Section 4.1).

There exists a bounded complex T^* of finitely generated R-modules with R-linear continuous $G_{F,S}$-actions that represents $\mathbf{R}\mathrm{Hom}_R(T, \omega_R)$ in the derived category of such modules. Nekovář's duality theorem is then as follows (see [Nek, Proposition 5.4.3(ii)]).[1]

Theorem (Nekovář)**.** *We have an isomorphism*

$$\mathbf{R}\Gamma(G_{F,S}, T) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_R\Big(\mathbf{R}\Gamma_{(c)}(G_{F,S}, T^*(1)), \omega_R\Big)[-3]$$

in the derived category of finitely generated R-modules.

In this paper, we consider a generalization of this to the setting of noncommutative Iwasawa theory. Suppose that F_∞ is a p-adic Lie extension of F contained in F_S. We denote by Γ the Galois group of the extension F_∞/F, and we let $\Lambda = R[\![\Gamma]\!]$ denote the resulting Iwasawa algebra over R. For a finitely generated R-module T with a continuous R-linear $G_{F,S}$-action, we define a finitely generated Λ-module $\mathscr{F}_\Gamma(T)$ with a continuous Λ-linear $G_{F,S}$-action by

$$\mathscr{F}_\Gamma(T) = \Lambda^\iota \otimes_R T,$$

where the superscript ι denotes that an element of Γ in Λ acts here on Λ by right multiplication by its inverse, and where $G_{F,S}$ acts on Λ through the quotient map $G_{F,S} \to \Gamma$ by left multiplication and then diagonally on the tensor product.

The $G_{F,S}$-cohomology of $\mathscr{F}_\Gamma(T)$ is interesting in that a version of Shapiro's lemma provides natural isomorphisms of continuous cohomology groups

$$H^n(G_{F,S}, \mathscr{F}_\Gamma(T)) \cong \varprojlim_\alpha H^n(G_{F_\alpha,S}, T)$$

[1] Actually, Nekovář's longer treatise is particularly concerned with a generalization of this duality that takes place between Selmer complexes, which we do not address in this article.

for every $n \geq 0$, where the limit is taken over α indexing the finite Galois extensions F_α of F that are contained in F_∞ (cf. [Lim2, Lemma 4.3.1]). That is, the cohomology groups of $\mathscr{F}_\Gamma(T)$ are the Iwasawa cohomology groups of the module T itself for the extension F_∞/F, and this identification is one of Λ-modules. We also have the analogous statements for compactly supported cohomology, as seen in [Lim2, Proposition 4.3.3]. Therefore, we can reduce the question of finding dualities among Iwasawa cohomology groups of compact modules to that of obtaining dualities among cohomology groups for $G_{F,S}$ itself. Note that, however, it is not even a priori clear in this setting that $H^n(G_{F,S}, \mathscr{F}_\Gamma(T))$ is a finitely generated Λ-module, let alone that we can find such a duality of Λ-modules.

The following is our main theorem (cf. Theorem 4.5.1).

Theorem. *We have isomorphisms*

$$\mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_\Gamma(T)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R^{\mathbf{L}} \omega_R\Big)[-3]$$
$$\mathbf{R}\Gamma_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R^{\mathbf{L}} \omega_R\Big)[-3]$$

in the derived category of finitely generated Λ-modules.

Nekovář proves the above theorem in the case that Γ is abelian [Nek, Theorem 8.5.6]. In fact, in that case, it is an almost immediate consequence of his above-mentioned theorem. That is, suppose for instance that $\Gamma \cong \mathbb{Z}_p^r$ for some $r \geq 1$. Then Λ is a complete commutative Noetherian local ring with finite residue field of characteristic p. Moreover, its dualizing complex is isomorphic to $\Lambda \otimes_R^{\mathbf{L}} \omega_R$ in the derived category of Λ-modules [Nek, Lemma 8.4.5.6]. Therefore, the commutative theory described above applies to Λ, and Nekovář is able to deduce the result from this. On the other hand, since we are working with nonabelian Γ and hence noncommutative Λ, we do not know that there exists a (nice enough) dualizing complex, and so the proof of our main theorem takes a different route. The idea is a simple one, though the proof is rather involved: after reducing to the case that R is $\mathbb{Z}_p$-flat, we perform an inductive argument, using the grading on Λ arising from the powers of its augmentation ideal, to deduce our result from Nekovář's.

We remark that, in their manuscript on the noncommutative main conjecture, Fukaya and Kato stated an analogue of our main theorem, with Λ replacing $\Lambda \otimes_R^{\mathbf{L}} \omega_R$ [FK, (1.6.12)], which in turn generalized a result of Burns and Flach [BF, Lemma 12(b)] for a narrower class of rings. The result of Fukaya-Kato applies to a more general class of (adic) rings Λ than ours and replaces $\mathscr{F}_\Gamma(T)$ and $\mathscr{F}_\Gamma(T^*)^\iota$ by a bounded complex of $\Lambda[G_{F,S}]$-modules X and its Λ-dual $\mathrm{Hom}_\Lambda(X, \Lambda)$. However, in order to be able to work with the Λ-dual, they assume that X consists of (finitely generated) projective Λ-modules. In the case of

Iwasawa cohomology that we study, the complex T need not be quasi-isomorphic to a bounded complex of $R[G_{F,S}]$-modules that are projective and finitely generated over R. Moreover, if R is a Gorenstein, then R serves as an R-dualizing complex, and our result reduces to a duality with respect to Λ itself, as in the result of Fukaya-Kato. Finally, we remark that Vauclair also proved a noncommutative duality theorem for induced modules in the case that $R = \mathbb{Z}_p$ and T is $\mathbb{Z}_p$-free, via a rather different method [Vau, Theorem 6.4].

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2 Preliminaries

In this section, we lay out some facts regarding modules over profinite rings that we shall require later in the paper. Elsewhere in the paper, we use some of the standard terminology and conventions regarding complexes, cones, shifts, total complexes of homomorphisms and tensor products, and derived categories, as can be found in [Nek], or in [Lim2] off of which this paper builds. In particular, all sign conventions will be as in the latter two papers. For further background on derived categories, we suggest the excellent book [Wei], as well as the more advanced text [KS].

2.1 Tensor products and homomorphism groups

In this subsection, R is a commutative ring. By an R-algebra, we will mean a ring with a given homomorphism from R to its center. In this section, we construct derived bifunctors of homomorphism groups and tensor products for bimodules over R-algebras and study a few isomorphisms that result from these constructions.

We use Λ° to denote the opposite ring to an R-algebra Λ. Given two R-algebras Λ and Σ, we are interested in a subclass of the class of Λ-Σ-bimodules, namely, the class of Λ-Σ-bimodules with the extra property that the left R-action coincides with the right R-action. We can (and shall) identify the category of such Λ-Σ-bimodules with the category

of $\Lambda \otimes_R \Sigma^\circ$-modules, and there are natural exact forgetful functors

$$\mathrm{res}_\Lambda \colon \mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ} \longrightarrow \mathrm{Mod}_\Lambda \quad \text{and} \quad \mathrm{res}_{\Sigma^\circ} \colon \mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ} \longrightarrow \mathrm{Mod}_{\Sigma^\circ},$$

which extend to exact functors on the derived categories. One observes that the categories $\mathrm{Mod}_{\Lambda\otimes_R R^\circ}$ and Mod_Λ are equivalent, as are $\mathrm{Mod}_{R\otimes_R\Sigma^\circ}$ and $\mathrm{Mod}_{\Sigma^\circ}$. We have following lemma, which extends that of [Ye, Lemma 2.1].

Lemma 2.1.1.

(*a*) *If Σ is a projective (resp., flat) R-algebra, then* res_Λ *preserves projective (resp., flat) modules. Similarly, if Λ is a projective (resp., flat) R-algebra, then* $\mathrm{res}_{\Sigma^\circ}$ *preserves projective (resp., flat) modules.*

(*b*) *If Σ is a flat R-algebra, then* res_Λ *preserves injectives. Similarly, if Λ is a flat R-algebra, then* $\mathrm{res}_{\Sigma^\circ}$ *preserves injectives.*

Proof. (a) We prove the statements of the first sentence, those of the second being a consequence. Suppose that Σ is a projective R-algebra. Since projective modules are the summands of free modules, it suffices to show that $\Lambda\otimes_R\Sigma^\circ$ is a projective Λ-module. Since Σ is a central R-algebra, we have $\Sigma \cong \Sigma^\circ$ as R-modules. Therefore, we have $\Sigma^\circ \oplus P \cong L$ for some projective R-module P and free R-module L. Then $\Lambda\otimes_R\Sigma^\circ$ is a direct summand of $\Lambda \otimes_R L$, which is a free Λ-module. Hence, $\Lambda \otimes_R \Sigma^\circ$ is a projective Λ-module.

Now suppose that Σ is flat over R. Since flat modules are direct limits of finitely generated free modules (see [Lam, Theorem 4.34]) and tensor products preserve direct limits, it suffices to show that $\Lambda \otimes_R \Sigma^\circ$ is a flat Λ-algebra. Since Σ is flat over R, we have that Σ is a direct limit of finitely generated free R-modules, which implies that $\Lambda \otimes_R \Sigma^\circ$ is a direct limit of finitely generated free Λ-modules.

(b) We shall prove this for res_Λ, the case of $\mathrm{res}_{\Sigma^\circ}$ being a consequence. The functor from Mod_Λ to $\mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ}$ sending M to $M \otimes_R \Sigma^\circ$ is exact by our assumption and is left adjoint to the functor res_Λ. The conclusion then follows from [Wei, Prop. 2.3.10]. $\square$

Let us briefly introduce the notions of q-projective and q-injective complexes of Λ-modules and several facts regarding them arising from the work of Spaltenstein, as can be found in [Spa], [Kel], [Lip, Section 2.3], and [KS, Chapter 14]. A complex P of Λ-modules is called q-projective if for every map $g\colon P \to B$ and quasi-isomorphism $s\colon A \to B$ of complexes of Λ-modules, there exists a map $f\colon P \to A$ of complexes of Λ-modules such that g and $s \circ f$ are homotopy equivalent. We remark that P is q-projective if and

only if it is homotopy equivalent to a direct limit of bounded above complexes P_n of projective Λ-modules via maps $\iota_n\colon P_n \to P_{n+1}$ that are split injective in each degree with quotients $P_{n+1}/\iota_n(P_n)$ having zero differentials. In particular, bounded above complexes of projectives are q-projective. If A is a complex of Λ-modules, then there exists a quasi-isomorphism $P \to A$ with P q-projective.

We also have the dual notion of q-injective complexes. A complex I of Λ-modules is q-injective if for every map $f\colon A \to I$ and quasi-isomorphism $s\colon A \to B$ of complexes of Λ-modules, there exists a map $g\colon B \to I$ such that f and $g \circ s$ are homotopy equivalent. A complex I of Λ-modules is q-injective if and only if it is homotopy equivalent to an inverse limit of bounded below complexes I_n of injective Λ-modules via maps $\pi_n\colon I_{n+1} \to I_n$ that are split surjective in each degree with kernels $\ker \pi_n$ having zero differentials. In particular, bounded below complexes of injective Λ-modules are q-injective. If A is a complex of Λ-modules, then there exists a quasi-isomorphism $A \to I$ with I q-injective.

In addition to Λ and Σ, we now let Ω be an R-algebra. If A is a $\Lambda \otimes_R \Omega^\circ$-module and B is a $\Lambda \otimes_R \Sigma^\circ$-module, we give $\mathrm{Hom}_\Lambda(A, B)$ the structure of an $\Omega \otimes_R \Sigma^\circ$-module via the left Ω and right Σ-actions

$$(\omega \cdot f)(a) = f(a\omega) \quad \text{and} \quad (f \cdot \sigma)(a) = f(a)\sigma$$

for $f \in \mathrm{Hom}_\Lambda(A, B)$, $a \in A$, $\omega \in \Omega$, and $\sigma \in \Sigma$.

Moreover, if A is a complex of $\Lambda \otimes_R \Omega^\circ$-modules and B is a complex of $\Lambda \otimes_R \Sigma^\circ$-modules, we define a complex $\mathrm{Hom}_\Lambda(A, B)$ of $\Omega \otimes_R \Sigma^\circ$-modules by

$$\mathrm{Hom}^n_\Lambda(A, B) = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_\Lambda(A^i, B^{i+n}),$$

with the usual differentials, as in [Lim2, Section 1].

Proposition 2.1.2. *Let A be a complex of $\Lambda \otimes_R \Omega^\circ$-modules, and let B be a complex of $\Lambda \otimes_R \Sigma^\circ$-modules. If Ω is a projective R-algebra or Σ is a flat R-algebra, then we have a derived bifunctor*

$$\mathbf{R}\mathrm{Hom}_\Lambda(-,-)\colon \mathbf{D}(\mathrm{Mod}_{\Lambda \otimes_R \Omega^\circ})^\circ \times \mathbf{D}(\mathrm{Mod}_{\Lambda \otimes_R \Sigma^\circ}) \longrightarrow \mathbf{D}(\mathrm{Mod}_{\Omega \otimes_R \Sigma^\circ}),$$

and $\mathrm{Hom}_\Lambda(A, B)$ represents $\mathbf{R}\mathrm{Hom}_\Lambda(A, B)$ if A is q-projective as a complex of Λ-modules and Ω is R-projective or B is q-injective as a complex of Λ-modules and Σ is R-flat.

Proof. Let us first assume that Ω is a projective R-algebra. Since every complex of $\Lambda \otimes_R \Omega^\circ$-modules is quasi-isomorphic to an q-projective complex of $\Lambda \otimes_R \Omega^\circ$-modules, we have a derived functor

$$\mathbf{R}\mathrm{Hom}_\Lambda(-, B)\colon \mathbf{D}(\mathrm{Mod}_{\Lambda \otimes_R \Omega^\circ})^\circ \longrightarrow \mathbf{D}(\mathrm{Mod}_{\Omega \otimes_R \Sigma^\circ})$$

(see [Lip, Corollary 2.3.2.3]). Suppose that $f\colon B \xrightarrow{\sim} B'$ is a quasi-isomorphism of complexes of $\Lambda \otimes_R \Sigma^\circ$-modules. Let $\varepsilon\colon P \xrightarrow{\sim} A$ be a quasi-isomorphism of complexes of $\Lambda \otimes_R \Omega^\circ$-modules, where P is q-projective. Then $\mathbf{R}\mathrm{Hom}_\Lambda(P, B)$ (resp., $\mathbf{R}\mathrm{Hom}_\Lambda(P, B')$) is represented by $\mathrm{Hom}_\Lambda(P, B)$ (resp., $\mathrm{Hom}_\Lambda(P, B')$). Applying Lemma 2.1.1(a), we see that P is also a q-projective complex of Λ-modules, so by [Lip, Proposition 2.3.8], the induced map

$$f_*\colon \mathrm{Hom}_\Lambda(P, B) \longrightarrow \mathrm{Hom}_\Lambda(P, B')$$

is a quasi-isomorphism of complexes of abelian groups. Since f_* is a morphism of complexes of $\Omega \otimes_R \Sigma^\circ$-modules, it is a quasi-isomorphism of such complexes. Hence, f induces isomorphisms

$$\mathbf{R}\mathrm{Hom}_\Lambda(A, B) \longrightarrow \mathbf{R}\mathrm{Hom}_\Lambda(A, B'),$$

proving the existence of the derived bifunctor. Moreover, if A is q-projective as a complex Λ-modules, then

$$\varepsilon_*\colon \mathrm{Hom}_\Lambda(A, B) \longrightarrow \mathrm{Hom}_\Lambda(P, B)$$

is a quasi-isomorphism of complexes of Σ°-modules, hence of $\Omega \otimes_R \Sigma^\circ$-modules as well. Thus, $\mathrm{Hom}_\Lambda(A, B)$ represents $\mathbf{R}\mathrm{Hom}_\Lambda(A, B)$, as desired.

If Σ is R-flat, the argument for the existence of the derived bifunctor and its computation by $\mathrm{Hom}_\Lambda(A, B)$ in the case that B is q-injective as a complex of Λ-modules is the direct analogue of that of the above argument, employing Lemma 2.1.1(b). To see that the resulting derived functor agrees with that contstructed above in the case that both Ω is R-projective and Σ is R-flat, consider a quasi-isomorphism $\varepsilon\colon P \xrightarrow{\sim} A$ as above and a quasi-isomorphism $\delta\colon B \xrightarrow{\sim} I$ with I a q-injective complex of $\Lambda \otimes_R \Sigma^\circ$-modules. As we have quasi-isomorphisms

$$\mathrm{Hom}_\Lambda(P, B) \xrightarrow{\delta_*} \mathrm{Hom}_\Lambda(P, I) \xleftarrow{\varepsilon_*} \mathrm{Hom}_\Lambda(A, I)$$

of complexes of $\Omega \otimes_R \Sigma^\circ$-modules, the derived functors coincide. $\square$

Note that one always has a map $\mathrm{Hom}_\Lambda(A, B) \to \mathbf{R}\mathrm{Hom}_\Lambda(A, B)$ (canonical up to isomorphism in the derived category), induced either by a quasi-isomorphism $P \xrightarrow{\sim} A$ with P q-projective or a quasi-isomorphism $B \xrightarrow{\sim} I$ with I q-injective.

Corollary 2.1.3. *If Ω is a projective R-algebra, then we have a commutative diagram*

$$\begin{tikzcd}
\mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Omega^\circ})^\circ \times \mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ}) \arrow[r, "\mathbf{R}\mathrm{Hom}_\Lambda(-,-)"] \arrow[d, "\mathrm{res}_\Lambda \times \mathrm{id}"'] & \mathbf{D}(\mathrm{Mod}_{\Omega\otimes_R\Sigma^\circ}) \arrow[d, "\mathrm{res}_{\Sigma^\circ}"] \\
\mathbf{D}(\mathrm{Mod}_\Lambda)^\circ \times \mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ}) \arrow[r, "\mathbf{R}\mathrm{Hom}_\Lambda(-,-)"] & \mathbf{D}(\mathrm{Mod}_{\Sigma^\circ}).
\end{tikzcd}$$

In the case that Σ is a flat R-algebra, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Omega^\circ})^\circ \times \mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ}) \arrow[r, "\mathbf{R}\mathrm{Hom}_\Lambda(-,-)"] \arrow[d, "\mathrm{id}\times\mathrm{res}_\Lambda"'] & \mathbf{D}(\mathrm{Mod}_{\Omega\otimes_R\Sigma^\circ}) \arrow[d, "\mathrm{res}_\Omega"] \\
\mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Omega^\circ})^\circ \times \mathbf{D}(\mathrm{Mod}_\Lambda) \arrow[r, "\mathbf{R}\mathrm{Hom}_\Lambda(-,-)"] & \mathbf{D}(\mathrm{Mod}_\Omega),
\end{tikzcd}$$

If A is a complex of $\Omega \otimes_R \Lambda^\circ$-modules and B is a complex of $\Lambda \otimes_R \Sigma^\circ$-modules, we define a complex $A \otimes_\Lambda B$ of $\Omega \otimes_R \Sigma^\circ$-modules by

$$(A \otimes_\Lambda B)^n = \bigoplus_{i\in\mathbb{Z}} A^i \otimes_\Lambda B^{n-i},$$

again with the usual differentials, as in [Lim2, Section 1].

We also have a notion of q-flat complexes of Λ°-modules (see [Lip, Section 2.5] and [Spa, Section 5] in the case of commutative rings, the proofs being identical); that is, a Λ°-module A is said to be q-flat if for every quasi-isomorphism $B \to C$ of Λ-modules, the resulting map $A \otimes_\Lambda B \to A \otimes_\Lambda C$ is also a quasi-isomorphism. In particular, q-projective complexes of Λ°-modules are q-flat, any bounded above complex of flat Λ°-modules is q-flat, and any filtered direct limit of q-flat complexes is q-flat. As in the case of homomorphism complexes, the total tensor product induces derived bifunctors as follows. We omit the analogous proof.

Proposition 2.1.4. *Let A and B be bounded below complexes of $\Omega \otimes_R \Lambda^\circ$-modules and $\Lambda \otimes_R \Sigma^\circ$-modules, respectively. If either Ω or Σ is a flat R-algebra, then we have a derived bifunctor*

$$- \otimes^{\mathbf{L}}_\Lambda - \colon \mathbf{D}(\mathrm{Mod}_{\Omega\otimes_R\Lambda^\circ}) \times \mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ}) \longrightarrow \mathbf{D}(\mathrm{Mod}_{\Omega\otimes_R\Sigma^\circ}),$$

and $A \otimes_\Lambda B$ represents $A \otimes^{\mathbf{L}}_\Lambda B$ if A is q-flat as a complex of Λ°-modules and Ω is R-flat or if B is q-flat as a complex of Λ-modules and Σ is R-flat.

Proposition 2.1.4 has the following direct corollary.

Corollary 2.1.5. *If Ω is a flat R-algebra, then we have a commutative diagram*

$$\begin{tikzcd}
\mathbf{D}(\mathrm{Mod}_{\Omega\otimes_R\Lambda^\circ}) \times \mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Sigma}) \arrow[r, "- \otimes^{\mathbf{L}}_\Lambda -"] \arrow[d, "\mathrm{res}_{\Lambda^\circ}\times\mathrm{id}"'] & \mathbf{D}(\mathrm{Mod}_{\Omega\otimes_R\Sigma^\circ}) \arrow[d, "\mathrm{res}_{\Sigma^\circ}"] \\
\mathbf{D}(\mathrm{Mod}_{\Lambda^\circ}) \times \mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ}) \arrow[r, "- \otimes^{\mathbf{L}}_\Lambda -"] & \mathbf{D}(\mathrm{Mod}_{\Sigma^\circ}),
\end{tikzcd}$$

and if Σ is a flat R-algebra, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{D}(\mathrm{Mod}_{\Omega\otimes_R\Lambda^\circ}) \times \mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Sigma}) \arrow[r, "- \otimes^{\mathbf{L}}_{\Lambda} -"] \arrow[d, "\mathrm{id}\times\mathrm{res}_\Lambda"'] & \mathbf{D}(\mathrm{Mod}_{\Omega\otimes_R\Sigma^\circ}) \arrow[d, "\mathrm{res}_\Omega"] \\
\mathbf{D}(\mathrm{Mod}_{\Omega\otimes_R\Lambda^\circ}) \times \mathbf{D}(\mathrm{Mod}_{\Lambda}) \arrow[r, "- \otimes^{\mathbf{L}}_{\Lambda} -"] & \mathbf{D}(\mathrm{Mod}_{\Omega}).
\end{tikzcd}$$

We end this section with some general lemmas regarding the passing of tensor products through homomorphism groups and the resulting isomorphisms in the derived categories. Let us use $\bar{m}$ to denote the degree of an element m of a term of a complex M of modules over a ring. We fix a fourth R-algebra Ξ.

Lemma 2.1.6. *Suppose that Ξ is a flat R-algebra and Σ is a projective R-algebra. Let A be a complex of $\Omega \otimes_R \Lambda^\circ$-modules, let B be a complex of $\Lambda \otimes_R \Sigma^\circ$-modules, and let C be a complex of $\Omega \otimes_R \Xi^\circ$-modules. Fix a quasi-isomorphism $P \to B$ of $\Lambda \otimes_R \Sigma^\circ$-modules with P q-projective over Λ and a quasi-isomorphism $C \to I$ of $\Omega \otimes_R \Xi^\circ$-modules with I q-injective over Ω. Then the adjunction isomorphism*

$$\begin{gathered}
\mathrm{Hom}_\Omega(A \otimes_\Lambda P, I) \longrightarrow \mathrm{Hom}_\Lambda(P, \mathrm{Hom}_\Omega(A, I)) \\
f \mapsto (p \mapsto (-1)^{\bar{a}\bar{p}} f(a \otimes p))
\end{gathered}$$

induces an isomorphism

$$\mathbf{R}\mathrm{Hom}_\Omega(A \otimes^{\mathbf{L}}_\Lambda B, C) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_\Lambda(B, \mathbf{R}\mathrm{Hom}_\Omega(A, C))$$

in $\mathbf{D}(\mathrm{Mod}_{\Sigma\otimes_R\Xi^\circ})$.

Proof. By Propositions 2.1.2 and 2.1.4, $\mathrm{Hom}_\Omega(A \otimes_\Lambda P, I)$ represents $\mathbf{R}\mathrm{Hom}_\Omega(A \otimes^{\mathbf{L}}_\Lambda B, C)$ and $\mathrm{Hom}_\Lambda(P, \mathrm{Hom}_\Omega(A, I))$ represents $\mathbf{R}\mathrm{Hom}_\Lambda(B, \mathbf{R}\mathrm{Hom}_\Omega(A, C))$. Therefore, we are reduced to showing that the adjunction map is an isomorphism of complexes, and this is standard (see [Lim2, Lemma 1.2]). □

We consider the next isomorphism first on the level of complexes.

Lemma 2.1.7. *Suppose that Ω is a flat R-algebra. Let A be a bounded complex of $\Omega\otimes_R\Sigma^\circ$-modules that are flat as Σ°-modules, let B be a complex of $\Xi\otimes_R\Lambda^\circ$-modules that are finitely presented as Λ°-modules, and let C be a complex of $\Sigma \otimes_R \Lambda^\circ$-modules. Suppose also that either the terms of A are finitely presented as Σ°-modules or at least one of B and C is bounded above and at least one is bounded below. Then the map*

$$\begin{gathered}
A \otimes_\Sigma \mathrm{Hom}_{\Lambda^\circ}(B, C) \longrightarrow \mathrm{Hom}_{\Lambda^\circ}(B, A \otimes_\Sigma C) \\
a \otimes f \mapsto \big(b \mapsto a \otimes f(b)\big)
\end{gathered}$$

is an isomorphism of complexes of $\Omega \otimes_R \Xi^\circ$*-modules.*

Proof. That the stated map is a map of complexes is an easy check of the actions and signs. To see that it is an isomorphism, consider first the case that A, B, and C are concentrated in degree 0 and take a presentation

$$F_1 \longrightarrow F_2 \longrightarrow B$$

with F_1 and F_2 finitely generated free Λ°-modules. Then the two rightmost vertical arrows in the commutative diagram with exact (noting the Σ°-flatness of A) rows

$$\begin{tikzcd}
0 \arrow[r] & A \otimes_\Sigma \mathrm{Hom}_{\Lambda^\circ}(B, C) \arrow[r] \arrow[d] & A \otimes_\Sigma \mathrm{Hom}_{\Lambda^\circ}(F_2, C) \arrow[r] \arrow[d] & A \otimes_\Sigma \mathrm{Hom}_{\Lambda^\circ}(F_1, C) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_{\Lambda^\circ}(B, A \otimes_\Sigma C) \arrow[r] & \mathrm{Hom}_{\Lambda^\circ}(F_2, A \otimes_\Sigma C) \arrow[r] & \mathrm{Hom}_{\Lambda^\circ}(F_1, A \otimes_\Sigma C)
\end{tikzcd}$$

are isomorphisms of Ω-modules by universal property of the direct sum and the commutativity of direct sums and tensor products, so the other is as well. Since the latter map is a morphism of complexes of $\Omega \otimes_R \Xi^\circ$-modules, we have the result in the case of modules.

In the general setting, we have that

$$\mathrm{Hom}^n_{\Lambda^\circ}(B, A \otimes_\Sigma C) = \prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\Lambda^\circ}\Big(B^j, \bigoplus_{i \in \mathbb{Z}} A^i \otimes_\Sigma C^{n-i+j}\Big).$$

Again since B^j is finitely presented over Λ°, a similar argument to that given above implies that the latter group is naturally isomorphic to

$$\prod_{j \in \mathbb{Z}} \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\Lambda^\circ}(B^j, A^i \otimes_\Sigma C^{n-i+j}).$$

On the other hand,

$$(A \otimes_\Sigma \mathrm{Hom}_{\Lambda^\circ}(B, C))^n = \bigoplus_{i \in \mathbb{Z}} A^i \otimes_\Sigma \Big(\prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\Lambda^\circ}(B^j, C^{n-i+j})\Big),$$

If the terms of A are finitely presented over Σ°, then can use a finite presentation of A^i to see that the latter term is naturally isomorphic to

$$\bigoplus_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} A^i \otimes_{\Sigma^\circ} \mathrm{Hom}_{\Lambda^\circ}(B^j, C^{n-i+j}),$$

and the boundedness of A allows us to commute the direct product and direct sum. Finally, if at least one of B and C is bounded above and at least one is bounded below, then the products over $j \in \mathbb{Z}$ involve only finitely many nonzero terms (again using that A is bounded), and the result follows as direct sums commute with tensor products and each other. $\square$

In the derived category, we have the following generalization of [Ven, Proposition 6.1].

Lemma 2.1.8. *Suppose that Ω is a flat R-algebra and Ξ is a projective R-algebra. Let A be a bounded above complex of $\Omega \otimes_R \Sigma^\circ$-modules, let B be a complex of $\Xi \otimes_R \Lambda^\circ$-modules, and let C be a bounded below complex of $\Sigma \otimes_R \Lambda^\circ$-modules. Suppose that there exists a quasi-isomorphism $Q \to A$ of $\Omega \otimes_R \Sigma^\circ$-modules, where Q is bounded and has terms that are flat as Σ°-modules. Suppose also that there is a quasi-isomorphism $P \to B$ of complexes of $\Xi \otimes_R \Lambda^\circ$-modules, where P is q-projective as a complex of Λ°-modules with terms that are finitely presented over Λ°. We assume also that either C is bounded, P is bounded above, or the terms of Q are finitely presented as Σ°-modules. Then the map*

$$Q \otimes_\Sigma \mathrm{Hom}_{\Lambda^\circ}(P, C) \longrightarrow \mathrm{Hom}_{\Lambda^\circ}(P, Q \otimes_\Sigma C)$$

gives rise to an isomorphism

$$A \otimes^{\mathbf{L}}_\Sigma \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(B, C) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(B, A \otimes^{\mathbf{L}}_\Sigma C)$$

in $\mathbf{D}(\mathrm{Mod}_{\Omega \otimes_R \Xi^\circ})$.

Proof. By Propositions 2.1.2 and 2.1.4, $Q \otimes_\Sigma \mathrm{Hom}_{\Lambda^\circ}(P, C)$ represents $A \otimes^{\mathbf{L}}_\Sigma \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(B, C)$ and $\mathrm{Hom}_{\Lambda^\circ}(P, Q \otimes_\Sigma C)$ represents $\mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(B, A \otimes^{\mathbf{L}}_\Sigma C)$. That the stated map is an isomorphism follows from Lemma 2.1.7. $\square$

Remark 2.1.9. For the purposes of this article, we require only derived homomorphism and tensor product functors that are bounded above or below on one side. On the other hand, in Section 4.5, we will have need to represent derived homomorphism complexes using q-projective resolutions in an unbounded first variable. Hence, we have given a general treatment.

2.2 Modules over group rings

In this subsection, we preserve the notation of Section 2.1, and in addition, let G be a group. We begin by extend the notions of derived bifunctors of homorphism groups and tensor products to incorporate an additional action of G.

If A is a complex of $\Lambda[G]$-modules and B is a complex of $(\Lambda \otimes_R \Sigma^\circ)[G]$-modules, we give $\mathrm{Hom}_\Lambda(A, B)$ the structure of a complex of $\Sigma^\circ[G]$-modules via the standard G-action

$$(g \cdot f)(a) = g \cdot f(g^{-1}a)$$

for $g \in G$, $a \in A$, and $f \in \mathrm{Hom}_\Lambda(A, B)$. The following is then a slight weakening of the natural analogue of Proposition 2.1.2, written in condensed form.

Proposition 2.2.1. *There is a derived bifunctor*

$$\mathbf{R}\mathrm{Hom}_\Lambda(-,-)\colon \mathbf{D}(\mathrm{Mod}_{\Lambda[G]})^\circ \times \mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ[G]}) \longrightarrow \mathbf{D}(\mathrm{Mod}_{\Sigma^\circ[G]}).$$

Moreover, $\mathbf{R}\mathrm{Hom}_\Lambda(A, B)$ *can be represented by* $\mathrm{Hom}_\Lambda(A, B)$ *if* A *is q-projective as a complex of* Λ*-modules or, if* Σ *is* R*-flat,* B *is q-injective as a complex of* Λ*-modules.*

Proof. Proposition 2.1.2 (with Ω replaced by $R[G]^\circ$ and Σ replaced by $\Sigma^\circ[G]$) implies the existence of derived bifunctors with the desired properties that take values in the category $\mathbf{D}(\mathrm{Mod}_{R[G]^\circ\otimes_R\Sigma^\circ[G]})$. There is a natural exact functor

$$\mathrm{Mod}_{R[G]^\circ\otimes_R\Sigma^\circ[G]} \longrightarrow \mathrm{Mod}_{\Sigma^\circ[G]}$$

taking M to the same Σ°-module with new G-action

$$g \cdot m = (g^{-1} \otimes g) \cdot m$$

for $g \in G$ and $m \in M$. This in turn induces a functor on derived categories, and composition of the above derived functor with this functor yields the result. □

Next, if A is a complex of $(\Omega \otimes_R \Lambda^\circ)[G]$-modules and B is a complex of $(\Lambda \otimes_R \Sigma^\circ)[G]$-modules, we give $A \otimes_\Lambda B$ the structure of a complex of $(\Omega \otimes_R \Sigma^\circ)[G]$-modules via the G-action defined by $g(a \otimes b) = ga \otimes gb$ for $a \in A$, $b \in B$, and $g \in G$. We state a very slight weakening (for brevity) of an analogue of Proposition 2.1.4.

Proposition 2.2.2. *If* Ω *or* Σ *is a flat* R*-algebra, then we have a derived bifunctor*

$$- \otimes^{\mathbf{L}}_\Lambda -\colon \mathbf{D}(\mathrm{Mod}_{\Omega\otimes_R\Lambda^\circ[G]}) \times \mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ[G]}) \longrightarrow \mathbf{D}(\mathrm{Mod}_{\Omega\otimes_R\Sigma^\circ[G]}).$$

Moreover, $A \otimes_\Lambda B$ *represents* $A \otimes^{\mathbf{L}}_\Lambda B$ *if either* Ω *is* R*-flat and* A *is a q-flat as a complex of* Λ°*-modules or* Σ *is* R*-flat and* B *is q-flat as a complex of* Λ*-modules.*

Proof. Proposition 2.1.4 with Ω replaced by $\Omega[G]$ and Σ° replaced by $\Sigma^\circ[G]$ yields the existence of a derived bifunctor

$$- \otimes^{\mathbf{L}}_{\Lambda} - \colon \mathbf{D}(\mathrm{Mod}_{\Omega\otimes_R\Lambda^\circ[G]}) \times \mathbf{D}(\mathrm{Mod}_{\Lambda\otimes_R\Sigma^\circ[G]}) \longrightarrow \mathbf{D}(\mathrm{Mod}_{\Omega[G]\otimes_R\Sigma^\circ[G]})$$

which has the desired properties. There is a natural exact functor

$$\mathrm{Mod}_{\Omega[G]\otimes_R\Sigma^\circ[G]} \longrightarrow \mathrm{Mod}_{(\Omega\otimes_R\Sigma^\circ)[G]}$$

that takes an module M to the same $\Omega \otimes_R \Sigma^\circ$-module with new G-action

$$g \cdot m = (g \otimes g) \cdot m$$

for $g \in G$ and $m \in M$, and the desired bifunctor is the resulting composition of derived functors. $\square$

Suppose now that Λ is an R-algebra and that we are given a homomorphism $\chi\colon G \to \mathrm{Aut}_{\Lambda^\circ}(\Lambda)$, allowing us to endow Λ with the structure of a $\Lambda^\circ[G]$-module. We denote the resulting module by ${}_\chi\Lambda$. (If χ is trivial, then we continue to write Λ for ${}_\chi\Lambda$.) We also have a map $\chi^{-1}\colon G \to \mathrm{Aut}_\Lambda(\Lambda)$ defined by $\chi^{-1}(g)(\lambda) = \lambda \cdot \chi(g^{-1})(1)$ for $g \in G$ and $\lambda \in \Lambda$, so the resulting object Λ_χ is a $\Lambda[G]$-module. The relationship between ${}_\chi\Lambda$ and Λ_χ can be expressed by the evaluation-at-1 maps

$$\mathrm{Hom}_\Lambda(\Lambda_\chi, \Lambda) \longrightarrow {}_\chi\Lambda \quad \text{and} \quad \mathrm{Hom}_{\Lambda^\circ}({}_\chi\Lambda, \Lambda) \longrightarrow \Lambda_\chi,$$

which are isomorphisms of $\Lambda^\circ[G]$ and $\Lambda[G]$-modules, respectively.

Lemma 2.2.3. *Suppose that Λ is a flat R-algebra, and let $\chi\colon G \to \mathrm{Aut}_{\Lambda^\circ}(\Lambda)$ be a homomorphism. For any bounded above complex A of $R[G]$-modules that are finitely presented over R and any bounded below complex B of R-modules, the two maps*

$$\theta\colon {}_\chi\Lambda \otimes_R \mathrm{Hom}_R(A, B) \longrightarrow \mathrm{Hom}_\Lambda(\Lambda_\chi \otimes_R A, \Lambda \otimes_R B)$$
$$\theta'\colon \Lambda_\chi \otimes_R \mathrm{Hom}_R(A, B) \longrightarrow \mathrm{Hom}_{\Lambda^\circ}({}_\chi\Lambda \otimes_R A, \Lambda \otimes_R B)$$

defined by $\theta(\lambda\otimes f)(\mu\otimes x) = \mu\lambda\otimes f(x)$ and $\theta'(\lambda\otimes f)(\mu\otimes x) = \lambda\mu\otimes f(x)$ are isomorphisms of complexes of $\Lambda^\circ[G]$-modules and of complexes of $\Lambda[G]$-modules, respectively.

Proof. We focus on the case of θ, as the other case follows from it. Since ${}_\chi\Lambda$ is R-flat, Lemma 2.1.7 (with $\Omega = \Lambda^\circ[G]$ and $\Xi = R[G]$) implies that the natural map

$${}_\chi\Lambda \otimes_R \mathrm{Hom}_R(A, B) \xrightarrow{\sim} \mathrm{Hom}_R(A, {}_\chi\Lambda \otimes_R B)$$

is an isomorphism of complexes of $\Lambda^\circ[G] \otimes_R R[G]$-modules if we take the actions of G independently on both sides of each tensor product, and hence of $\Lambda^\circ[G]$-modules if we take the diagonal actions. The inverse of the isomorphism provided by the evaluation map provides the first of the isomorphisms of complexes of $\Lambda^\circ[G]$-modules

$$ {}_\chi\Lambda \otimes_R B \xrightarrow{\sim} \mathrm{Hom}_\Lambda(\Lambda_\chi, \Lambda) \otimes_R B \xrightarrow{\sim} \mathrm{Hom}_\Lambda(\Lambda_\chi, \Lambda \otimes_R B), $$

the canonical second map being an isomorphism as Λ_χ is free of rank 1 over Λ. We therefore have an isomorphism

$$ \mathrm{Hom}_R(A, {}_\chi\Lambda \otimes_R B) \xrightarrow{\sim} \mathrm{Hom}_R(A, \mathrm{Hom}_\Lambda(\Lambda_\chi, \Lambda \otimes_R B)). $$

Finally, the inverse of the adjoint morphism is an isomorphism

$$ \mathrm{Hom}_R(A, \mathrm{Hom}_\Lambda(\Lambda_\chi, \Lambda \otimes_R B)) \xrightarrow{\sim} \mathrm{Hom}_\Lambda(\Lambda_\chi \otimes_R A, \Lambda \otimes_R B) $$

of complexes of $\Lambda^\circ[G]$-modules, and the resulting composite of three isomorphisms is easily computed to be θ. $\square$

2.3 Projective modules over a profinite ring

Let Λ be a profinite ring, and fix a directed fundamental system $\mathcal{I}$ of open neighborhoods of zero consisting of two-sided ideals of Λ. We say that a topological Λ-module M is endowed with the $\mathcal{I}$-adic topology if the collection $\{\mathfrak{A}M\}_{\mathfrak{A}\in\mathcal{I}}$ forms a fundamental system of neighborhoods of zero. It was shown in [Lim2, Section 2.1] that any finitely generated compact (Hausdorff) Λ-module necessarily has the $\mathcal{I}$-adic topology, and, moreover, any homomorphism between such modules is necessarily continuous.

In this subsection, we recall several facts about projective Λ-modules that will be of use to us. We denote the abelian category of compact Λ-modules by $\mathcal{C}_\Lambda$. The free profinite Λ-module on a set X is canonically isomorphic to the topological direct product of one copy of Λ for each element of X [Wil, Proposition 7.4.1], and a profinite Λ-module P is projective if and only if it is continuously isomorphic to a direct summand of the free profinite module on a set of generators of P [Wil, Proposition 7.4.7]. In particular, the category $\mathcal{C}_\Lambda$ has enough projectives. Any projective object in $\mathcal{C}_\Lambda$ that is finitely generated over Λ is a projective Λ-module, and conversely, any finitely generated projective Λ-module endowed with the $\mathcal{I}$-adic topology is a projective object in $\mathcal{C}_\Lambda$ [Lim2, Proposition 2.1.8].

Recall that the projective dimension of an abstract Λ-module M is the minimum integer n (if it exists) such that there is a resolution of M by projective Λ-modules

$$0 \longrightarrow P^{-n} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow M \longrightarrow 0.$$

The topological projective dimension of a compact Λ-module is defined similarly, replacing projective Λ-modules by projective objects in $\mathcal{C}_\Lambda$. The notions of projective dimension and topological projective dimension coincide on finitely generated (compact) Λ-modules.

For the remainder of the subsection, we suppose that Λ is left Noetherian. Note that any projective resolution P of a finitely generated Λ-module M is quasi-isomorphic to a projective resolution Q of M by finitely generated Λ-modules via a map $Q \to P$ compatible with the augmentations to M. In particular, the projective dimension of M is the length of its shortest resolution by projectives in the category of finitely generated Λ-modules.

In general, if $\mathfrak{C}$ is a category with objects that are Λ-modules, we will use $\mathfrak{C}^{\Lambda-\mathrm{ft}}$ to denote the full subcategory of objects that are finitely generated (i.e., of finite type) over Λ. Since $\mathrm{Mod}_\Lambda^{\Lambda-\mathrm{ft}}$ has enough projectives, the equivalent category $\mathcal{C}_\Lambda^{\Lambda-\mathrm{ft}}$ also has enough projectives. Therefore, we may speak of the bounded above derived categories $\mathbf{D}^-(\mathrm{Mod}_\Lambda^{\Lambda-\mathrm{ft}})$ and $\mathbf{D}^-(\mathcal{C}_\Lambda^{\Lambda-\mathrm{ft}})$. Denote by $\mathbf{D}^-_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$ the full subcategory of $\mathbf{D}^-(\mathrm{Mod}_\Lambda)$ which has as its objects those bounded above complexes X of Λ-modules for which all of the $H^i(X)$ are finitely generated Λ-modules. The following standard lemma tells us that any such complex is quasi-isomorphic to a complex of finitely generated modules. (See [Nek, Proposition 3.2.6] for an analogous statement, which has a similar proof.)

Lemma 2.3.1. *Let Ω be a left Noetherian ring. Every bounded above complex X of Ω-modules for which every $H^i(X)$ is a finitely generated Ω-module has a quasi-isomorphic subcomplex of finitely generated Ω-modules.*

In sum, we have equivalences of categories

$$\mathbf{D}^-(\mathcal{C}_\Lambda^{\Lambda-\mathrm{ft}}) \xrightarrow{\sim} \mathbf{D}^-(\mathrm{Mod}_\Lambda^{\Lambda-\mathrm{ft}}) \xrightarrow{\sim} \mathbf{D}^-_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda),$$

the first being induced by the forgetful functor and the second by the inclusion of categories $\mathrm{Mod}_\Lambda^{\Lambda-\mathrm{ft}} \hookrightarrow \mathrm{Mod}_\Lambda$. We use these equivalences to identify these categories with each other.

2.4 Completed tensor products

In this subsection, we review some basic facts about completed tensor products and briefly study their derived functors. Let R be a commutative profinite ring, and let Λ, Ω, and

Σ be profinite R-algebras, by which we shall mean that they are profinite and the maps from R to their centers are continuous.

Let $\mathcal{I}$ (resp., $\mathcal{J}$) denote a directed fundamental system of open neighborhoods of zero consisting of two-sided ideals of Λ (resp., Ω). We then define a completed tensor product algebra by

$$\Omega \,\hat{\otimes}_R\, \Lambda^\circ = \varprojlim_{\mathfrak{A}\in\mathcal{I},\ \mathfrak{B}\in\mathcal{J}} \Omega/\mathfrak{B} \otimes_R (\Lambda/\mathfrak{A})^\circ.$$

This is clearly a profinite R-algebra.

We shall denote the category of compact $\Omega \,\hat{\otimes}_R\, \Lambda^\circ$-modules by $\mathcal{C}_{\Omega-\Lambda}$. Let M be an object of $\mathcal{C}_{\Omega-\Lambda}$, and let N be an object of $\mathcal{C}_{\Lambda-\Sigma}$. By [RZ, Lemma 5.1.1(a)], we have that the set of finite index $\Omega \,\hat{\otimes}_R\, \Lambda^\circ$-submodules (resp., $\Lambda \,\hat{\otimes}_R\, \Sigma^\circ$-submodules) of M (resp., N) forms a basis of neighborhoods of zero on M (resp., N). We define the completed tensor product to be the compact $\Omega \,\hat{\otimes}_R\, \Sigma^\circ$-module

$$M \,\hat{\otimes}_\Lambda\, N = \varprojlim_{M',N'} M/M' \otimes_\Lambda N/N',$$

where M' and N' run through the respective bases for M and N, with the topology defined by the inverse limit. The completed tensor product is associative and commutative (i.e., $M \,\hat{\otimes}_\Lambda\, N \cong N \,\hat{\otimes}_{\Lambda^\circ}\, M$) in the same sense as the usual tensor product.

Note that the canonical Λ-balanced map

$$t\colon M \times N \longrightarrow M \,\hat{\otimes}_\Lambda\, N$$

induces a homomorphism

$$M \otimes_\Lambda N \longrightarrow M \,\hat{\otimes}_\Lambda\, N$$

of $\Omega \otimes_R \Sigma^\circ$-modules that has dense image. The completed tensor product of M and N then satisfies the following universal property (see [RZ, Section 5.5] in the case that $\Omega = \Sigma = R$).

Lemma 2.4.1. *For any compact $\Omega \,\hat{\otimes}_R\, \Sigma^\circ$-module L and any continuous, Λ-balanced, bilinear map $f\colon M \times N \to L$ of $\Omega \otimes_R \Sigma^\circ$-modules, there is a unique continuous map $\hat{f}\colon M \,\hat{\otimes}_\Lambda\, N \to L$ such that $\hat{f} \circ t = f$.*

It follows that, in defining the completed tensor product, it suffices to run through a basis of neighborhoods of zero consisting of open $\Omega \,\hat{\otimes}_R\, \Lambda^\circ$-submodules of M and a basis of neighborhoods of zero consisting of open $\Lambda \,\hat{\otimes}_R\, \Sigma^\circ$-submodules of N.

The following is also standard.

Lemma 2.4.2. *Let M and N be objects of $\mathcal{C}_{\Omega-\Lambda}$ and $\mathcal{C}_{\Lambda-\Sigma}$, respectively.*

(*a*) *Suppose that $M = \varprojlim M_\alpha$ and $N = \varprojlim N_\beta$, where each M_α (resp., N_β) is a compact $\Omega \hat{\otimes}_R \Lambda^\circ$-module (resp., compact $\Lambda \hat{\otimes}_R \Sigma^\circ$-module). Then there is an isomorphism*

$$M \hat{\otimes}_\Lambda N \cong \varprojlim_{\alpha,\beta} M_\alpha \hat{\otimes}_\Lambda N_\beta$$

of compact $\Omega \hat{\otimes}_R \Sigma^\circ$-modules.

(*b*) *The canonical map $M \otimes_\Lambda N \to M \hat{\otimes}_\Lambda N$ is an isomorphism if either M is finitely generated as a Λ°-module or N is finitely generated as a Λ-module.*

(*c*) *The functor*

$$M \hat{\otimes}_\Lambda - \colon \mathcal{C}_{\Lambda-\Sigma} \longrightarrow \mathcal{C}_{\Omega-\Sigma}$$

is right exact.

The next lemma describes connections between projective objects in different categories of compact modules and flat objects in categories of abstract modules. A ring is said to be left (resp., right) coherent if all its finitely generated left (resp., right) ideals are finitely presented.

Lemma 2.4.3.

(*a*) *If Λ is left (resp., right) coherent, then every projective object in $\mathcal{C}_{\Lambda^\circ}$ (resp., $\mathcal{C}_\Lambda$) is flat with respect to the usual left (resp., right) tensor product over Λ.*

(*b*) *If Ω (resp., Λ) is a projective object of $\mathcal{C}_R$, then every projective object in $\mathcal{C}_{\Omega-\Lambda}$ is a projective object in $\mathcal{C}_{\Lambda^\circ}$ (resp., $\mathcal{C}_\Omega$).*

Proof. (a) Since every projective object in $\mathcal{C}_{\Lambda^\circ}$ is continuously isomorphic to a direct summand of a direct product of copies of Λ°, it suffices to show that any direct product of copies of Λ is flat as an abstract Λ°-module. By a theorem of Chase (cf. [Lam, Theorem 4.47]), this is equivalent to the fact that Λ is left coherent.

(b) Suppose that Ω is projective in $\mathcal{C}_R$, so Ω is continuously isomorphic to a direct summand of a direct product of copies of R. Lemma 2.4.2(a) then tells us that $\Omega \hat{\otimes}_R \Lambda^\circ$ is topologically a direct summand of a direct product of copies of Λ°, hence is projective. $\square$

Let G be a profinite group. We use $\mathcal{C}_{\Lambda,G}$ to denote the category of compact Λ-modules with a continuous commuting action of G, the morphisms being continuous homomorphisms of $\Lambda[G]$-modules. Note the following.

Remark 2.4.4. The category $\mathcal{C}_{\Lambda,G}$ is equivalent to the category $\mathcal{C}_{\Lambda[\![G]\!]}$, where $\Lambda[\![G]\!]$ is given the profinite topology defined by

$$\Lambda[\![G]\!] \cong \varprojlim_{\mathfrak{A}\in\mathcal{I}} \varprojlim_{N} (\Lambda/\mathfrak{A})[G/N],$$

where N runs over the open normal subgroups of G.

Consequently, $\mathcal{C}_{\Lambda,G}$ is an abelian category with enough projectives, and every element of $\mathcal{C}_{\Lambda,G}$ is the inverse limit of its finite $\Lambda[G]$-quotients (see also [Lim2, Section 2.2]). Moreover, $\Lambda[\![G]\!]$ is a projective object of $\mathcal{C}_\Lambda$ by [RZ, Lemma 5.3.5(d)], so the forgetful functor $\mathcal{C}_{\Lambda,G} \to \mathcal{C}_\Lambda$ takes projective objects to projective objects. To shorten notation, we use $\mathcal{C}_{\Omega-\Lambda,G}$ to denote the category $\mathcal{C}_{\Omega\,\hat{\otimes}_R\,\Lambda^\circ,G}$.

We remark that if M and N are, respectively, objects of $\mathcal{C}_{\Omega-\Lambda,G}$ and $\mathcal{C}_{\Lambda-\Sigma,G}$, possibly with trivial G-actions, we may give $M\,\hat{\otimes}_\Lambda\,N$ the structure of an object of $\mathcal{C}_{\Omega-\Sigma,G}$ via the diagonal action of G. That is, the G-action is defined by choosing bases of open $(\Omega\,\hat{\otimes}_R\,\Lambda^\circ)[G]$-submodules of M (resp., $(\Lambda\,\hat{\otimes}_R\,\Sigma^\circ)[G]$-submodules of N) and taking the inverse limits of the tensor products of the finite quotients.

Remark 2.4.5. The analogous result to Lemma 2.4.2 holds, as a consequence of said lemma, if we take M and N to be objects of $\mathcal{C}_{\Omega-\Lambda,G}$ and $\mathcal{C}_{\Lambda-\Sigma,G}$, respectively. (That is, in part (a), one must take M_α and N_β to be objects of these categories to attain an isomorphism in $\mathcal{C}_{\Omega-\Sigma,G}$, and in part (c), the functor is now a functor from $\mathcal{C}_{\Lambda-\Sigma,G}$ to $\mathcal{C}_{\Omega-\Sigma,G}$.)

Note that we may form completed tensor products of complexes as with the usual tensor products.

Proposition 2.4.6. *If Ω is projective in $\mathcal{C}_R$, then the completed tensor product induces the following derived bifunctor*

$$-\,\hat{\otimes}^{\mathbf{L}}_\Lambda\,-\colon \mathbf{D}^-(\mathcal{C}_{\Omega-\Lambda,G}) \times \mathbf{D}^-(\mathcal{C}_{\Lambda-\Sigma,G}) \longrightarrow \mathbf{D}^-(\mathcal{C}_{\Omega-\Sigma,G}),$$

where $A\,\hat{\otimes}_\Lambda\,B$ represents $A\,\hat{\otimes}^{\mathbf{L}}_\Lambda\,B$ if the terms of A are projective as objects in $\mathcal{C}_{\Lambda^\circ}$. Furthermore, there is a commutative diagram

$$\begin{tikzcd}
\mathbf{D}^-(\mathcal{C}_{\Omega-\Lambda,G}) \times \mathbf{D}^-(\mathcal{C}_{\Lambda-\Sigma,G}) \arrow[r, "-\,\hat{\otimes}^{\mathbf{L}}_\Lambda\,-"] \arrow[d] & \mathbf{D}^-(\mathcal{C}_{\Omega-\Sigma,G}) \arrow[d] \\
\mathbf{D}^-(\mathcal{C}_{\Lambda^\circ,G}) \times \mathbf{D}^-(\mathcal{C}_{\Lambda-\Sigma,G}) \arrow[r, "-\,\hat{\otimes}^{\mathbf{L}}_\Lambda\,-"] & \mathbf{D}^-(\mathcal{C}_{\Sigma^\circ,G})
\end{tikzcd}$$

in which the vertical arrows are induced by forgetful functors, and if Λ is left Noetherian, there is a commutative diagram

$$\begin{tikzcd}
\mathbf{D}^-(\mathcal{C}_{\Omega-\Lambda,G}) \times \mathbf{D}^-(\mathcal{C}_{\Lambda-\Sigma,G}) \arrow[r, "-\hat{\otimes}^{\mathbf{L}}_{\Lambda}-"] & \mathbf{D}^-(\mathcal{C}_{\Omega-\Sigma,G}) \arrow[d] \\
\mathbf{D}^-(\mathcal{C}_{\Omega-\Lambda,G}) \times \mathbf{D}^-(\mathcal{C}^{\Lambda-\mathrm{ft}}_{\Lambda-\Sigma,G}) \arrow[u] \arrow[d] & \\
\mathbf{D}^-(\mathrm{Mod}_{(\Omega\otimes_R\Lambda^\circ)[G]}) \times \mathbf{D}^-(\mathrm{Mod}_{(\Lambda\otimes_R\Sigma^\circ)[G]}) \arrow[r, "-\otimes^{\mathbf{L}}_{\Lambda}-"] & \mathbf{D}^-(\mathrm{Mod}_{(\Omega\otimes_R\Sigma^\circ)[G]})
\end{tikzcd}$$

in which the vertical arrows are induced by forgetful functors and embeddings of categories.

Proof. The proof of the first part and the commutativity of the first set of diagrams follow by similar arguments to those of Propositions 2.1.3(a) and 2.2.2, making use of Lemma 2.4.3(b) (using bounded above complexes of projective objects in place of q-projective complexes). The commutativity of the final diagram then follows from Lemmas 2.4.2(b) and 2.4.3. □

2.5 Ind-admissible modules

The notion of an ind-admissible $R[G]$-module was introduced in [Nek, Section 3.3] for a complete commutative Noetherian local ring R with finite residue field and a profinite group G. An $R[G]$-module is ind-admissible if it can be written as a union of $R[G]$-submodules that are finitely generated over R and on which G acts continuously with respect to the topology defined by the maximal ideal of R. In this section, we discuss an analogous construction of ind-admissible modules over noncommutative profinite rings.

As in Nekovář's treatment, we do not consider the seemingly delicate issue of placing topologies on ind-admissible modules, as it proves unnecessary. In particular, it is still possible to define the continuous cochain complex of an ind-admissible module as a direct limit. Like the category of discrete modules, the category of ind-admissible modules has enough injectives, so one can take the derived functor of the cochain functor on bounded below complexes of ind-admissible modules, something one can only do for (cohomologically) bounded complexes of compact modules.

We maintain the notation of Section 2.4. Moreover, we suppose that Ω is left Noetherian and Λ is right Noetherian. For an $(\Omega \otimes_R \Lambda^\circ)[G]$-module M, we denote by $\mathcal{S}(M)$ the set of $(\Omega \otimes_R \Lambda^\circ)[G]$-submodules of M that are finitely generated as Λ°-modules and on

which Ω and G act continuously with respect to the $\mathcal{I}$-adic topology. The following is a straightforward generalization of [Nek, Lemma 3.3.2].

Lemma 2.5.1. *Let M be an $(\Omega \otimes_R \Lambda^\circ)[G]$-module.*

(a) *If $M' \in \mathcal{S}(M)$, then $N \in \mathcal{S}(M)$ for every $(\Omega \otimes_R \Lambda^\circ)[G]$-submodule N of M'.*

(b) *If $f\colon M \to N$ is a homomorphism of $(\Omega \otimes_R \Lambda^\circ)[G]$-modules and $M' \in \mathcal{S}(M)$, then $f(M') \in \mathcal{S}(N)$.*

(c) *If M', $M'' \in \mathcal{S}(M)$, then $M' + M'' \in \mathcal{S}(M)$.*

Proof. For part (a), Corollary 2.1.6 and Proposition 2.1.7 of [Lim2] imply that the subspace topology on N from the $\mathcal{I}$-adic topology on M agrees with the $\mathcal{I}$-adic topology on N, which implies that Ω and G act continuously on N. In (b), the continuity of the Ω and G-actions on the finitely generated Λ°-module $f(M')$ is a consequence of the fact that the map $M' \to f(M')$ is a continuous quotient map with respect to the $\mathcal{I}$-adic topology (as follows from Corollary 2.1.5 and Proposition 2.1.7 of [Lim2]). Part (c) follows from (b), using the addition map $M \times M \to M$. $\square$

Note that Lemma 2.5.1(c) implies that $\mathcal{S}(M)$ is a directed set with respect to inclusion. We say that a $(\Omega \hat{\otimes}_R \Lambda^\circ)[G]$-module M is (right) ind-admissible if

$$M = \bigcup_{N \in \mathcal{S}(M)} N.$$

We list some basic properties of the full subcategory $\mathscr{I}_{\Omega-\Lambda,G}$ of $\mathrm{Mod}_{(\Omega\otimes_R\Lambda^\circ)[G]}$ with objects the ind-admissible $(\Omega \otimes_R \Lambda^\circ)[G]$-modules.

Lemma 2.5.2.

(a) *The category $\mathscr{I}_{\Omega-\Lambda,G}$ is abelian and stable under subobjects, quotients and colimits.*

(b) *The embedding functor*

$$i\colon \mathscr{I}_{\Omega-\Lambda,G} \hookrightarrow \mathrm{Mod}_{(\Omega\otimes_R\Lambda^\circ)[G]}$$

is exact and is left adjoint to the functor

$$j\colon \mathrm{Mod}_{(\Omega\otimes_R\Lambda^\circ)[G]} \to \mathscr{I}_{\Omega-\Lambda,G}$$

that takes a module M to the union of the elements of $\mathcal{S}(M)$.

(c) *The category $\mathscr{I}_{\Omega-\Lambda,G}$ has enough injectives.*

(d) *Let M be an ind-admissible $(\Omega \otimes_R \Lambda^\circ)[G]$-module, and let N be a finitely generated Λ°-submodule of M. Then $(\Omega \otimes_R \Lambda^\circ)[G] \cdot N$ is an ind-admissible $(\Omega \otimes_R \Lambda^\circ)[G]$-module which is a finitely generated Λ°-module.*

(e) *Let M be a $(\Omega \otimes_R \Lambda^\circ)[G]$-module. Then $M \in \mathcal{S}(M)$ if and only if M is an ind-admissible $(\Omega \otimes_R \Lambda^\circ)[G]$-module which is finitely generated as a Λ°-module.*

Proof. For parts (a), (b) and (c), similar arguments to those of [Nek, Proposition 3.3.5] apply. The "only if" direction of (e) is obvious. For (d), since N is Λ°-finitely generated, we can find a finite subset $\{M_1, \ldots, M_n\}$ of $\mathcal{S}(M)$ such that

$$N \subseteq M_1 + \cdots + M_n.$$

The assertion then follows from Lemma 2.5.1(c) and the "only if" direction of (e). The "if" direction of (e) follows from (d), since $M = (\Omega \otimes_R \Lambda^\circ)[G] \cdot M$. □

If G is trivial, we write $\mathscr{I}_{\Omega-\Lambda}$ for $\mathscr{I}_{\Omega-\Lambda,G}$. We leave to the reader the proof of the following.

Lemma 2.5.3.

(a) *Let M be an $(\Omega \otimes_R \Lambda^\circ)[G]$-module. If N is an $(\Omega \otimes_R \Lambda^\circ)[G]$-submodule of M, then $N \in \mathcal{S}(M)$ if and only if N is an object of $\mathcal{C}^{\Lambda^\circ-\mathrm{ft}}_{\Omega-\Lambda,G}$.*

(b) *The category $\mathscr{I}_{\Omega-\Lambda,G}$ is equivalent to the ind-category of $\mathcal{C}^{\Lambda^\circ-\mathrm{ft}}_{\Omega-\Lambda,G}$.*

(c) *The category $\mathscr{I}_{\Omega-\Lambda}$ is equivalent to the full subcategory of $\mathscr{I}_{\Omega-\Lambda,G}$ with objects the modules on which G acts trivially.*

Remark 2.5.4. In the case that $\Omega = R$, we denote $\mathscr{I}_{R-\Lambda,G}$ by $\mathscr{I}_{\Lambda^\circ,G}$, and the latter category is equivalent to the ind-category of $\mathcal{C}^{\Lambda^\circ-\mathrm{ft}}_{\Lambda^\circ,G}$. The subcategory $\mathscr{I}_{\Lambda^\circ}$ is simply $\mathrm{Mod}_{\Lambda^\circ}$.

Lemma 2.5.5. *Let A be an object of $\mathcal{C}^{\Omega-\mathrm{ft}}_{\Omega,G}$, and let B be an object of $\mathcal{C}^{\Lambda^\circ-\mathrm{ft}}_{\Omega-\Lambda,G}$. Then $\mathrm{Hom}_\Omega(A, B)$ with the $\mathcal{I}$-adic topology is an object of $\mathcal{C}^{\Lambda^\circ-\mathrm{ft}}_{\Lambda^\circ,G}$.*

Proof. Since A is finitely generated over Ω and B is a compact Ω-module, we have by [Lim2, Lemma 2.1.4(3)] that $\mathrm{Hom}_\Omega(A, B) = \mathrm{Hom}_{\Omega,\mathrm{cts}}(A, B)$, where the latter group is the group of continuous homomorphism of Ω-modules. Note that Λ° acts continuously on $\mathrm{Hom}_\Omega(A, B)$ with respect to the compact-open topology by [F, Proposition 3(a)],

and similarly G acts continuously on it as a consequence of [F, Lemma 2]. We have a continuous isomorphism of $\Lambda^\circ[G]$-modules

$$\mathrm{Hom}_\Omega(A,B) \xrightarrow{\sim} \varprojlim_\beta \mathrm{Hom}_\Omega(A,B_\beta),$$

where B_β runs over the finite $(\Omega \hat{\otimes}_R \Lambda^\circ)[G]$-quotients of B. As B_β is finite and therefore discrete, the group $\mathrm{Hom}_\Omega(A,B_\beta)$ is finite and discrete as well, so $\mathrm{Hom}_\Omega(A,B)$ is compact.

Note that $\mathrm{Hom}_\Omega(A,B)$ injects into $\mathrm{Hom}_\Omega(\Omega^r,B) \cong B^r$ for some $r \geq 0$, since A is Ω-finitely generated. We therefore have that $\mathrm{Hom}_\Omega(A,B)$ is Λ°-finitely generated. Finally, [Lim2, Proposition 2.1.7] then implies that the compact-open topology on $\mathrm{Hom}_\Omega(A,B)$ agrees with the $\mathcal{I}$-adic topology. $\square$

We define

$$\mathrm{Hom}_{\Omega,\mathrm{cts}}(-,-)\colon \mathcal{C}_{\Omega,G} \times \mathscr{I}_{\Omega-\Lambda,G} \longrightarrow \mathscr{I}_{\Lambda^\circ,G}.$$

by

$$\mathrm{Hom}_{\Omega,\mathrm{cts}}(A,B) = \varinjlim_\alpha \varinjlim_\beta \mathrm{Hom}_\Omega(A_\alpha,B_\beta),$$

with $A = \varprojlim A_\alpha$, where A_α runs over the quotients of A in $\mathcal{C}_{\Omega,G}^{\Omega-\mathrm{ft}}$, and $B = \varinjlim B_\beta$, where B_β runs over the elements of $\mathcal{S}(B)$, i.e., by Lemma 2.5.3, the subobjects of B that lie in $\mathcal{C}_{\Omega-\Lambda,G}^{\Lambda^\circ-\mathrm{ft}}$. Note that it actually suffices to let A_α and B_β run over cofinal subsets.

Remark 2.5.6. We note that

$$\mathrm{Hom}_{\Omega,\mathrm{cts}}(A,B) \cong \varinjlim_\beta \mathrm{Hom}_{\Omega,\mathrm{cts}}(A,B_\beta),$$

where $\mathrm{Hom}_{\Omega,\mathrm{cts}}(A,B_\beta)$ here is the $(\Omega \otimes_R \Lambda^\circ)[G]$-module of continuous Ω-module homomorphisms in the usual sense.

We also note the following.

Lemma 2.5.7. *Let A be an object of $\mathcal{C}_{\Omega,G}^{\Omega-\mathrm{ft}}$, and let B be an object of $\mathscr{I}_{\Omega-\Lambda,G}$. Then we have*

$$\mathrm{Hom}_{\Omega,\mathrm{cts}}(A,B) = \mathrm{Hom}_\Omega(A,B).$$

Proof. This follows from the computation

$$\mathrm{Hom}_{\Omega,\mathrm{cts}}(A,B) \cong \varinjlim_\beta \mathrm{Hom}_{\Omega,\mathrm{cts}}(A,B_\beta) = \varinjlim_\beta \mathrm{Hom}_\Omega(A,B_\beta) \cong \mathrm{Hom}_\Omega(A,B),$$

the latter isomorphism following from the case in which A is free of finite rank, since A is finitely presented over Ω. $\square$

We may extend the definition of modules of continuous Ω-homomorphisms to complexes.

Proposition 2.5.8. *There is a derived bifunctor*

$$\mathbf{R}\mathrm{Hom}_{\Omega,\mathrm{cts}}(-,-)\colon \mathbf{D}^-(\mathcal{C}_{\Omega,G})^\circ \times \mathbf{D}(\mathscr{I}_{\Omega-\Lambda,G}) \longrightarrow \mathbf{D}(\mathscr{I}_{\Lambda^\circ,G}),$$

and $\mathbf{R}\mathrm{Hom}_{\Omega,\mathrm{cts}}(A,B)$ *can be represented by* $\mathrm{Hom}_{\Omega,\mathrm{cts}}(A,B)$ *if the terms of* A *are projective as objects of* $\mathcal{C}_\Omega$*. Moreover, we have a commutative diagram*

$$\begin{tikzcd}
\mathbf{D}^-(\mathcal{C}_{\Omega,G})^\circ \times \mathbf{D}^+(\mathscr{I}_{\Omega-\Lambda,G}) \arrow[r, "\mathbf{R}\mathrm{Hom}_{\Omega,\mathrm{cts}}(-,-)"] & \mathbf{D}(\mathscr{I}_{\Lambda^\circ,G}) \arrow[dd] \\
\mathbf{D}^-(\mathcal{C}_{\Omega,G}^{\Omega-\mathrm{ft}})^\circ \times \mathbf{D}^+(\mathscr{I}_{\Omega-\Lambda,G}) \arrow[u] \arrow[d] & \\
\mathbf{D}^-(\mathrm{Mod}_{\Omega[G]})^\circ \times \mathbf{D}^+(\mathrm{Mod}_{(\Omega\otimes_R\Lambda^\circ)[G]}) \arrow[r, "\mathbf{R}\mathrm{Hom}_{\Omega}(-,-)"] & \mathbf{D}(\mathrm{Mod}_{\Lambda^\circ[G]})
\end{tikzcd}$$

in which the vertical arrows arise from forgetful functors and the natural embeddings of abelian categories.

Proof. The existence of the derived bifunctor in question follows from an argument similar to that of Proposition 2.1.2, and the second statement is a consequence of the fact that a projective object in $\mathcal{C}_{\Omega,G}$ restricts to a projective object in $\mathcal{C}_\Omega$, noting Remark 2.5.6.

Let A be a bounded above complex of objects of $\mathcal{C}_{\Omega,G}^{\Omega-\mathrm{ft}}$, and let B be a complex of objects of $\mathscr{I}_{\Omega-\Lambda,G}$. Choose bounded above complexes L, P and Q, consisting of projective objects in $\mathcal{C}_{\Omega,G}$, $\mathrm{Mod}_{\Omega[G]}$ and $\mathrm{Mod}_\Omega^{\Omega-\mathrm{ft}}$ respectively, with quasi-isomorphisms to A of complexes in the respective categories. Projectivity yields natural maps $Q \to L$, $Q \to P$ and $P \to L$ of complexes in $\mathcal{C}_\Omega$, Mod_Ω and $\mathrm{Mod}_{\Omega[G]}$, respectively. We then have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\Omega,\mathrm{cts}}(L,B) \arrow[r] \arrow[d] & \mathrm{Hom}_{\Omega,\mathrm{cts}}(Q,B) \arrow[d] \\
\mathrm{Hom}_\Omega(P,B) \arrow[r] & \mathrm{Hom}_\Omega(Q,B).
\end{tikzcd}$$

Here, the righthand vertical arrow is the identity map by Lemma 2.5.7, and the lower horizontal map is a quasi-isomorphism as a consequence of Proposition 2.2.1, as $\Omega[G]$ is Ω-free. We want to show that the left hand vertical arrow is a quasi-isomorphism of complexes of $\Lambda^\circ[G]$-modules, and we will be done if we can show the upper horizontal arrow is a quasi-isomorphism of complexes of Λ°-modules.

In the case that B is a module, exactness of direct limit reduces us to the case that B is an object of $\mathcal{C}^{\Lambda^\circ-\mathrm{ft}}_{\Omega-\Lambda,G}$, hence of $\mathcal{C}_\Omega$. In this case, since both L and Q are complexes of projectives in $\mathcal{C}_\Omega$, we obtain that the map is a quasi-isomorphism.

In the general setting, since L and Q are bounded above and B is bounded below, we have

$$\mathrm{Hom}^n_{\Omega,\mathrm{cts}}(X,B) = \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\Omega,\mathrm{cts}}(X^j, B^{j+n})$$

for $X = L$ and $X = Q$, and the direct sum commutes with direct limits. As B is the direct limit of its truncations $\tau_{\leq i}B$, we may therefore assume that B is bounded with $B^j = 0$ for $j > i$. In this case, brutal truncations provide an exact triangle

$$(\sigma_{\leq i-1}B)[i-1] \longrightarrow B^i \longrightarrow (\sigma_{\leq i}B)[i] \longrightarrow (\sigma_{\leq i-1}B)[i],$$

and we are reduced recursively to the above-proven case of a module. □

The following analogue of Lemma 2.2.3 will be of later use.

Lemma 2.5.9. *Let A be a bounded above complex of objects in $\mathcal{C}_{R,G}$, and let B be a bounded below complex of R-modules. Then there are isomorphisms*

$$\theta\colon {}_\chi\Lambda \otimes_R \mathrm{Hom}_{R,\mathrm{cts}}(A,B) \longrightarrow \mathrm{Hom}_{\Lambda,\mathrm{cts}}(\Lambda_\chi \hat{\otimes}_R A, \Lambda \otimes_R B)$$
$$\theta'\colon \Lambda_\chi \otimes_R \mathrm{Hom}_{R,\mathrm{cts}}(A,B) \longrightarrow \mathrm{Hom}_{\Lambda^\circ,\mathrm{cts}}({}_\chi\Lambda \hat{\otimes}_R A, \Lambda \otimes_R B)$$

that arise as direct limits of the maps in Lemma 2.2.3.

Proof. Write $A = \varprojlim A_\alpha$ with A_α an object of $\mathcal{C}^{R-\mathrm{ft}}_{R,G}$ and $B = \varinjlim B_\beta$ with B_β an object of $\mathcal{C}^{R-\mathrm{ft}}_R$. Note that

$$\Lambda_\chi \hat{\otimes}_R A \cong \varprojlim_\alpha (\Lambda_\chi \otimes_R A_\alpha),$$

so, applying Lemma 2.2.3, we have

$$\begin{aligned}\mathrm{Hom}_{\Lambda,\mathrm{cts}}(\Lambda_\chi \hat{\otimes}_R A, \Lambda \otimes_R B) &= \varinjlim_{\alpha,\beta} \mathrm{Hom}_\Lambda(\Lambda_\chi \otimes_R A_\alpha, \Lambda \otimes_R B_\beta)\\ &\cong \varinjlim_{\alpha,\beta} {}_\chi\Lambda \otimes_R \mathrm{Hom}_R(A_\alpha, B_\beta)\\ &\cong {}_\chi\Lambda \otimes_R \mathrm{Hom}_{R,\mathrm{cts}}(A,B).\end{aligned}$$

This yields the first isomorphism, and the argument for the second is similar. □

Let M be an ind-admissible $(\Omega \otimes_R \Lambda^\circ)[G]$-module. The continuous cochain complex of degree $i \geq 0$ on G with values in M is the complex of $\Omega \otimes_R \Lambda^\circ$-modules that is the direct limit of complexes

$$C(G, M) = \varinjlim_{N \in \mathcal{S}(M)} C(G, N),$$

where $C(G, N)$ is the usual complex of (inhomogeneous) continuous cochains, with N given the $\mathcal{I}$-adic topology.

We remark that, in this definition, it clearly suffices to take the direct limit over a cofinal subset of $\mathcal{S}(M)$. In particular, if M itself is finitely generated over Λ°, then by Lemma 2.5.2(e) the above definition of $C(G, M)$ as a direct limit agrees with its definition as continuous cochains, considering M as an object of $\mathcal{C}_{\Omega-\Lambda,G}$. Finally, we can extend the above definition in the usual manner to consider the total cochain complex of a complex of ind-admissible modules, as in [Nek, (3.4.1.3)], and as $\mathscr{I}_{\Omega-\Lambda,G}$ has enough injectives, we have a derived functor

$$\mathbf{R}\Gamma(G, -) \colon \mathbf{D}^+(\mathscr{I}_{\Omega-\Lambda,G}) \longrightarrow \mathbf{D}^+(\mathrm{Mod}_{\Omega\otimes_R\Lambda^\circ}).$$

3 Cohomology groups of induced modules

In this paper, we are interested in the case that our profinite ring Λ is the Iwasawa algebra of a compact p-adic Lie group. With this in mind, we introduce some notation that will be used from this point forward. Fix a prime p. Let R be a commutative complete Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k, where k is finite of characteristic p. We let Γ denote a compact p-adic Lie group. We are interested in the completed group ring $\Lambda = R[\![\Gamma]\!]$.

We note that Λ is a profinite ring, endowed with the topology given by the canonical isomorphism

$$\Lambda \xrightarrow{\sim} \varprojlim_{U \in \mathcal{U}} \varprojlim_{n \geq 1} R/\mathfrak{m}^n[\Gamma/U],$$

where $\mathcal{U}$ denotes the set of open normal subgroups of Γ. In fact, it is a projective object in $\mathcal{C}_R$ (cf. [RZ, Lemma 5.3.5(d)]). Moreover, we have the following.

Proposition 3.0.1. *The ring Λ is Noetherian.*

Proof. As R is a complete local Noetherian ring with finite residue field k, the Cohen structure theorem [Coh, Theorem 12] implies that it is isomorphic to a quotient of a power series ring S in $n = \dim_k \mathfrak{m}/\mathfrak{m}^2$ variables over the ring of Witt vectors $\mathcal{O}$ of k.

Therefore, $R[\![\Gamma]\!]$ is a quotient of $S[\![\Gamma]\!]$, so it suffices to prove that $S[\![\Gamma]\!]$ is Noetherian. Since $S[\![\Gamma]\!] \cong \mathcal{O}[\![\mathbb{Z}_p^n \times \Gamma]\!]$ and $\mathbb{Z}_p^n \times \Gamma$ is a compact p-adic Lie group, we have that $S[\![\Gamma]\!]$ is Noetherian by a mild extension of a classical theorem of Lazard's (cf. [Ven2, Corollary 2.4] and [Laz, Proposition V.2.2.4]). $\square$

Since Noetherian rings are necessarily coherent, we may apply Lemma 2.4.3(a) to conclude from Proposition 3.0.1 that Λ is a flat R-algebra.

3.1 Induced modules and descent

In this subsection, we will introduce induced modules over an Iwasawa algebra. We then exhibit an interesting spectral sequence relating the cohomology of the induced modules to the cohomology of the modules themselves.

Let G be a profinite group, and fix a continuous homomorphism $\chi\colon G \to \Gamma$ of profinite groups. Since Γ may be viewed as a subgroup of $\mathrm{Aut}_{\Lambda^\circ}(\Lambda)$ by left multiplication, we may define the $\Lambda^\circ[G]$-module ${}_\chi\Lambda$ as in Section 2.2. If A is a $\Lambda \otimes_R \Lambda^\circ$-module, we then set ${}_\chi A = {}_\chi\Lambda \otimes_\Lambda A$ and $A_\chi = A \otimes_\Lambda \Lambda_\chi$, which are $\Lambda^\circ[G]$ and $\Lambda[G]$-modules, respectively.

Let $\mathscr{U}$ denote the set of open normal subgroups of Γ, and let $U \in \mathscr{U}$. Since $R[\Gamma/U]$ is a quotient of Λ, the ring Λ acts on it through the quotient map by left and right multiplication. Moreover, G also acts on $R[\Gamma/U]$ by left multiplication, after composing χ with the quotient map from Γ to Γ/U. This G-action commutes with the right Λ-action.

Let M be a $R[G]$-module. As in [Lim2, Section 4.1], we define a $\Lambda[G]$-module $\mathscr{F}_\Gamma(M)$ by

$$\mathscr{F}_\Gamma(M) = \varprojlim_{U \in \mathscr{U}} (R[\Gamma/U]_\chi \otimes_R M)$$

with G acting diagonally and Λ acting on the left on the terms in the inverse limit. This construction applied to compact modules provides a functor

$$\mathscr{F}_\Gamma\colon \mathcal{C}_{R,G} \longrightarrow \mathcal{C}_{\Lambda,G},$$

which is exact as $R[\Gamma/U]$ is R-flat and the inverse limit is exact on inverse systems in $\mathcal{C}_{\Lambda,G}$. We extend this to a functor on complexes in the usual fashion.

Given an complex T of objects in $\mathcal{C}_{R,G}$, we have, noting Lemma 2.4.2(a), a natural continuous isomorphism

$$\mathscr{F}_\Gamma(T) \cong \Lambda_\chi \hat{\otimes}_R T$$

of complexes in $\mathcal{C}_{\Lambda,G}$. We may therefore use Lemma 2.4.2(b) to note that $\mathscr{F}_\Gamma$ takes a complex T of objects in in $\mathcal{C}_{R,G}^{R-\mathrm{ft}}$ to the complex $\Lambda_\chi \otimes_R T$ of objects in $\mathcal{C}_{\Lambda,G}^{\Lambda-\mathrm{ft}}$. We make the latter identification freely.

Fix a profinite R-algebra Ω that is a projective object of $\mathcal{C}_R$ for the remainder of the section. We are now able to supply a key ingredient for descent.

Lemma 3.1.1. *Let T be a bounded complex in $\mathcal{C}_{R,G}$. Let Γ' be a quotient of Γ by a closed subgroup, and set $\Lambda' = R[\![\Gamma']\!]$. Let N be a bounded above complex of objects of $\mathcal{C}_{\Omega-\Lambda'}$. Then N can viewed as a complex of objects in $\mathcal{C}_{\Omega-\Lambda}$ via the quotient map $\Lambda \twoheadrightarrow \Lambda'$, and this map induces an isomorphism*

$$N \hat{\otimes}^{\mathbf{L}}_{\Lambda} \mathscr{F}_{\Gamma}(T) \xrightarrow{\sim} N \hat{\otimes}^{\mathbf{L}}_{\Lambda'} \mathscr{F}_{\Gamma'}(T)$$

in $\mathbf{D}^-(\mathcal{C}_{\Omega,G})$.

Proof. We use Proposition 2.4.6 repeatedly. Note that $N \hat{\otimes}_{\Lambda} \Lambda_{\chi}$ represents $N \hat{\otimes}^{\mathbf{L}}_{\Lambda} \Lambda_{\chi}$ in $\mathbf{D}^-(\mathcal{C}_{\Omega-\Lambda,G})$, and similarly with Λ' replacing Λ. As the quotient map $\Lambda \to \Lambda'$ induces an isomorphism

$$N \hat{\otimes}_{\Lambda} \Lambda_{\chi} \xrightarrow{\sim} N \hat{\otimes}_{\Lambda'} \Lambda'_{\chi},$$

we obtain an isomorphism

$$(N \hat{\otimes}^{\mathbf{L}}_{\Lambda} \Lambda_{\chi}) \hat{\otimes}^{\mathbf{L}}_{R} T \xrightarrow{\sim} (N \hat{\otimes}^{\mathbf{L}}_{\Lambda'} \Lambda'_{\chi}) \hat{\otimes}^{\mathbf{L}}_{R} T$$

in $\mathbf{D}^-(\mathcal{C}_{\Omega,G})$. These derived completed tensor products are seen to be associative using projective resolutions in the first and third variables. Since Λ_{χ} and Λ'_{χ} are projective in $\mathcal{C}_R$, we also have

$$N \hat{\otimes}^{\mathbf{L}}_{\Lambda} \mathscr{F}_{\Gamma}(T) \cong N \hat{\otimes}^{\mathbf{L}}_{\Lambda} (\Lambda_{\chi} \hat{\otimes}^{\mathbf{L}}_{R} T) \quad \text{and} \quad N \hat{\otimes}^{\mathbf{L}}_{\Lambda'} \mathscr{F}_{\Gamma'}(T) \cong N \hat{\otimes}^{\mathbf{L}}_{\Lambda} (\Lambda'_{\chi} \hat{\otimes}^{\mathbf{L}}_{R} T),$$

and combining these isomorphisms yields the result. □

The following lemma will also be of use in the case of interest to us. We remark that we may, and will often, identify a bounded derived category (in either or both directions) with the larger but equivalent full subcategory of the unbounded derived category which has as its objects the complexes with bounded cohomology.

Lemma 3.1.2. *Let T be a bounded complex in $\mathcal{C}_{R,G}$. Let N be a bounded above complex of objects of $\mathcal{C}^{\Lambda^\circ-\mathrm{ft}}_{\Omega-\Lambda}$, and suppose that N is quasi-isomorphic to a bounded complex of projective Λ°-modules. Then $N \hat{\otimes}^{\mathbf{L}}_{\Lambda} \mathscr{F}_{\Gamma}(T)$ lies in $\mathbf{D}^b(\mathcal{C}_{\Omega,G})$.*

Proof. Let Q be a bounded complex of finitely generated projective Λ°-modules equipped with a quasi-isomorphism to N. Then $Q \otimes_{\Lambda} \mathscr{F}_{\Gamma}(T)$ represents $N \hat{\otimes}^{\mathbf{L}}_{\Lambda} \mathscr{F}_{\Gamma}(T)$ in $\mathbf{D}^b(\mathcal{C}_{R,G})$, noting Lemma 2.4.2(b). Since Ω is projective in $\mathcal{C}_R$, the first commutative diagram of Proposition 2.4.6 (with $\Sigma = R$) yields the result. □

For a cohomologically bounded complex M of objects in $\mathcal{C}_{\Lambda,G}$, we let $\mathbf{R}\Gamma(G,M)$ denote the corresponding derived object in $\mathbf{D}(\mathrm{Mod}_\Lambda)$ attached to its (total) complex $C(G,M)$ of continuous G-cochains, and we let $H^i(G,M)$ denote its ith continuous G-cohomology (or, more precisely, hypercohomology) group. We recall from [Lim2, Proposition 2.2.11] that

$$\mathbf{R}\Gamma(G,-)\colon \mathbf{D}^b(\mathcal{C}_{\Lambda,G}) \longrightarrow \mathbf{D}^+(\mathrm{Mod}_\Lambda)$$

is an exact functor.

Theorem 3.1.3. *Let T be a bounded complex in $\mathcal{C}_{R,G}$. Let Γ' be a quotient of Γ by a closed subgroup, and set $\Lambda' = R[\![\Gamma']\!]$. Let N be a bounded complex of objects of $\mathcal{C}^{(\Lambda')^\circ-\mathrm{ft}}_{\Omega-\Lambda'}$ that have finite projective dimension over $(\Lambda')^\circ$. Viewing N as a compact Λ°-module via the augmentation map $\Lambda \to \Lambda'$, we have an isomorphism*

$$N \otimes^{\mathbf{L}}_\Lambda \mathbf{R}\Gamma(G,\mathscr{F}_\Gamma(T)) \xrightarrow{\sim} \mathbf{R}\Gamma(G, N \hat{\otimes}^{\mathbf{L}}_{\Lambda'} \mathscr{F}_{\Gamma'}(T))$$

in $\mathbf{D}(\mathrm{Mod}_\Omega)$.

Proof. Let P be a bounded above complex of projective objects in $\mathcal{C}_{\Omega-\Lambda}$ mapping quasi-isomorphically to N. Then the terms of P are flat Λ°-modules by Lemma 2.4.3, so it follows from Proposition 2.1.4 that $P \otimes_\Lambda C(G,\mathscr{F}_\Gamma(T))$ represents $N \otimes^{\mathbf{L}}_\Lambda \mathbf{R}\Gamma(G,\mathscr{F}_\Gamma(T))$ in $\mathbf{D}(\mathrm{Mod}_\Omega)$. Moreover, it follows from Lemmas 3.1.1 and 3.1.2 that $P \hat{\otimes}_\Lambda \mathscr{F}_\Gamma(T)$ is cohomologically bounded and represents $N \hat{\otimes}^{\mathbf{L}}_{\Lambda'} \mathscr{F}_{\Gamma'}(T)$ in $\mathbf{D}(\mathcal{C}_{\Omega,G})$. Noting the universality of the completed tensor product of Lemma 2.4.1, we have a map of complexes of Ω-modules

$$P \otimes_\Lambda C(G,\mathscr{F}_\Gamma(T)) \longrightarrow C(G, P \hat{\otimes}_\Lambda \mathscr{F}_\Gamma(T))$$

with sign conventions as in [Nek, Proposition 3.4.4]. It suffices to show that this map is a quasi-isomorphism.

Let Q be a bounded above complex of finitely generated projective Λ°-modules mapping quasi-isomorphically to N. We then have a map of complexes of R-modules

$$Q \otimes_\Lambda C(G,\mathscr{F}_\Gamma(T)) \xrightarrow{\sim} C(G, Q \otimes_\Lambda \mathscr{F}_\Gamma(T))$$

that, much as in [Nek, Proposition 3.4.4], is an isomorphism. (To see that it is an isomorphism, note that it is immediate if Q is a finitely generated free Λ°-module, and therefore also for direct summands of such, i.e., the finitely generated projective Λ°-modules. The case of a complex is then immediate.) By the projectivity of Q in both the categories of

abstract and compact Λ°-modules, we have a commutative diagram

$$\begin{array}{ccc} Q \otimes_\Lambda C(G, \mathscr{F}_\Gamma(T)) & \xrightarrow{\sim} & C(G, Q \otimes_\Lambda \mathscr{F}_\Gamma(T)) \\ \downarrow & & \downarrow \\ P \otimes_\Lambda C(G, \mathscr{F}_\Gamma(T)) & \longrightarrow & C(G, P \hat{\otimes}_\Lambda \mathscr{F}_\Gamma(T)) \end{array}$$

in which the vertical arrows are quasi-isomorphisms of complexes of R-modules, noting Lemma 2.4.2(b). It follows that the lower horizontal arrow is a quasi-isomorphism as well. $\square$

We make a couple of remarks.

Remark 3.1.4. Suppose that G is a finite group. The proof of Theorem 3.1.3 then goes through word-for-word with Tate cochains and the resulting derived complexes replacing the usual cochains of G and its derived complexes.

Remark 3.1.5. The proof of Theorem 3.1.3 extends easily to show that

$$N \otimes_\Lambda^{\mathbf{L}} \mathbf{R}\Gamma(G, Y) \xrightarrow{\sim} \mathbf{R}\Gamma(G, N \hat{\otimes}_\Lambda^{\mathbf{L}} Y)$$

for Ω any profinite R-algebra that is a projective object in $\mathcal{C}_R$, Λ any left coherent and right Noetherian profinite R-algebra, Y any bounded complex in $\mathcal{C}_{\Lambda,G}$, and N any bounded above complex of objects of $\mathcal{C}_{\Omega-\Lambda}^{\Lambda^\circ-\mathrm{ft}}$ such that $N \hat{\otimes}_\Lambda^{\mathbf{L}} Y$ has bounded cohomology. In the special case that Y is an object of $\mathcal{C}_{\Lambda,G}^{\Lambda-\mathrm{ft}}$ that is projective as a Λ-module, this more general result is in turn a special case of [FK, Proposition 1.6.5(3)], which holds without the assumptions on Λ.

We point out the following special case of the theorem, which allows us to descend.

Corollary 3.1.6. *Let T be a bounded complex in $\mathcal{C}_{R,G}$. Let Γ' be quotient of Γ by a closed subgroup, and set $\Lambda' = R[\![\Gamma']\!]$. Then we have an isomorphism*

$$\Lambda' \otimes_\Lambda^{\mathbf{L}} \mathbf{R}\Gamma(G, \mathscr{F}_\Gamma(T)) \xrightarrow{\sim} \mathbf{R}\Gamma(G, \mathscr{F}_{\Gamma'}(T))$$

in $\mathbf{D}(\mathrm{Mod}_{\Lambda'})$.

3.2 Finite generation of cohomology groups

In this subsection, we shall show that the cohomology groups of induced modules are finitely generated under a certain assumption on the group. We maintain the notation of the previous subsection. Since Γ is a compact p-adic Lie group, it has an open normal pro-p subgroup. Let us fix such a subgroup U and set $\Lambda' = R[\Gamma/U]$. Note that the kernel of the canonical projection $\Lambda \twoheadrightarrow \Lambda'$ is precisely $I(U)\Lambda$, where $I(U)$ denotes the augmentation ideal of $R[\![U]\!]$. We are now able to prove the next lemma.

Lemma 3.2.1. *Let M be a compact Λ-module. Then M is finitely generated over Λ if and only if $\Lambda' \otimes_\Lambda M$ is finitely generated over R.*

Proof. Write $M_U = M/I(U)M$. We begin by noting that

$$\Lambda' \otimes_\Lambda M \cong \Lambda' \otimes_\Lambda M_U \cong \Lambda' \otimes_{\Lambda'} M_U \cong M_U,$$

so it suffices to show that M is finitely generated over Λ if and only if M_U is finitely generated over R. Since U is pro-p, the Jacobson radical $\mathfrak{M}(U)$ of $R[\![U]\!]$ is $\mathfrak{m}R[\![U]\!] + I(U)$ (see [NSW, Proposition 5.2.16(iii)]). This implies that

$$M/\mathfrak{M}(U)M \cong M_U/\mathfrak{m}M_U.$$

Therefore, Nakayama's lemma tells us that M_U is finitely generated over R if and only if $M/\mathfrak{M}(U)M$ is finite. On the other hand, Nakayama's lemma for compact Λ-modules (cf. [NSW, Lemma 5.2.18(ii)]) tells us that M is finitely generated over Λ if and only if $M/\mathfrak{M}(U)M$ is finite. $\square$

Lemma 3.2.1 requires a compact Λ-module. Therefore, we give a sufficient condition for an abstract Λ-module to be a compact Λ-module under an appropriate topology.

Lemma 3.2.2. *Suppose that M is an abstract Λ-module which is the inverse limit of an inverse system of finite quotient modules. Then M is a compact Λ-module with respect to the resulting profinite topology.*

Proof. We need only to show that the Λ-action

$$\theta\colon \Lambda \times M \longrightarrow M$$

is continuous with respect to the topology given by the inverse limit. Suppose first that M is finite. Let $\mathfrak{M}$ denote the Jacobson radical of Λ. Then $\mathfrak{M}^n M$ stabilizes, and it follows from Nakayama's lemma that we have $\mathfrak{M}^n M = 0$ for big enough n. By [NSW, Corollary

5.2.19], there exist $r \geq 0$ and $U \in \mathscr{U}$ such that $\mathfrak{m}^r\Lambda + I(U) \subseteq \mathfrak{M}^n$. Hence, θ is continuous with respect to the discrete topology on M.

In general, we can write $M \cong \varprojlim M/M_\alpha$, where $\{M_\alpha\}$ is a directed system of Λ-submodules of finite index. Let $(\lambda, x) \in \theta^{-1}(y + M_\alpha)$ for $\lambda \in \Lambda$ and $x, y \in M$. Since M/M_α is finite, it follows from the above discussion that there exist r and U such that

$$(\lambda + \mathfrak{m}^r\Lambda + I(U)) \cdot (x + M_\alpha) \subseteq y + M_\alpha,$$

as desired. □

We record another lemma which we will require later.

Lemma 3.2.3. *If M is a finitely generated Λ-module, then $\mathrm{Tor}_i^\Lambda(\Lambda', M)$ is finitely generated over R for every $i \geq 0$.*

Proof. To see this, we first choose a resolution P of M consisting of finitely generated projective Λ-modules. Then $\Lambda' \otimes_\Lambda P$ is a complex of finitely generated R-modules, as Λ' is a free R-module of finite rank. Therefore, the homology groups $\mathrm{Tor}_i^\Lambda(\Lambda', M)$ of the latter complex are finitely generated over R. □

We recall from [Nek, Proposition 4.2.3] that if G is a profinite group such that $H^i(G, M)$ is finite for every finite G-module M of p-power order and $i \geq 0$, then we have that $H^i(G, T)$ is a finitely generated R-module for every $T \in \mathcal{C}_{R,G}^{R-\mathrm{ft}}$ and $i \geq 0$. We will refer to a profinite group G as p-cohomologically finite if G is of finite p-cohomological dimension and has the property that the $H^i(G, M)$ are finite for every finite G-module M of p-power order.

Theorem 3.2.4. *Suppose that G is p-cohomologically finite. Let T be a bounded complex of objects in $\mathcal{C}_{R,G}^{R-\mathrm{ft}}$. Then the cohomology groups $H^i(G, \mathscr{F}_\Gamma(T))$ are finitely generated over Λ for all $i \geq 0$. That is, $\mathbf{R}\Gamma(G, \mathscr{F}_\Gamma(T))$ is an object of $\mathbf{D}^b_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$.*

Proof. Suppose for now that T is concentrated in degree 0. Corollary 3.1.6 provides the following bounded convergent spectral sequence

$$E_2^{r,s} = \mathrm{Tor}_{-r}^\Lambda(\Lambda', H^s(G, \mathscr{F}_\Gamma(T))) \Rightarrow H^{r+s}(G, \mathscr{F}_{\Gamma'}(T)),$$

where $\Gamma' = \Gamma/U$. Let n denote the cohomological dimension of G.

Let $j \leq n$, and suppose inductively that $H^i(G, \mathscr{F}_\Gamma(T))$ is a finitely generated Λ-module for all $i > j$. Since G has finite p-cohomological dimension, the spectral sequence is bounded, and it follows that $E_m^{r,s}$ must stabilize for large enough m. In particular, we

have that $E_\infty^{0,j}$ is a quotient of $E_2^{0,j}$. Note that $\mathscr{F}_{\Gamma'}(T)$ is finitely generated over R, as Γ' is finite, so $H^j(G, \mathscr{F}_{\Gamma'}(T))$ is a finitely generated R-module by [Nek, Proposition 4.2.3]. Since $E_\infty^{0,j}$ is a subquotient of $H^j(G, \mathscr{F}_{\Gamma'}(T))$, it is also a finitely generated R-module. On the other hand, it follows from the definition of $E_\infty^{0,j}$ that the kernel of the surjective map $E_2^{0,j} \to E_\infty^{0,j}$ is isomorphic to a subquotient of

$$\bigoplus_{j<i\leq n} E_2^{j-i-1,i} = \bigoplus_{j<i\leq n} \mathrm{Tor}_{i-j+1}^{\Lambda}(\Lambda', H^i(G, \mathscr{F}_\Gamma(T))).$$

By our induction hypothesis and Lemma 3.2.3, the above module is a finitely generated R-module. It follows that

$$E_2^{0,j} \cong \Lambda' \otimes_\Lambda H^j(G, \mathscr{F}_\Gamma(T))$$

is a finitely generated R-module. As $H^j(G, \mathscr{F}_\Gamma(T))$ is an inverse limit of finite Λ-modules by [Lim2, Proposition 2.2.8], Lemma 3.2.2 gives it the structure of a compact Λ-module, and we may therefore apply Lemma 3.2.1 to conclude that $H^j(G, \mathscr{F}_\Gamma(T))$ is finitely generated over Λ.

Now suppose more generally that T is a bounded complex. As in [Nek, (3.5.4.2)], we have the following convergent spectral sequence of Λ-modules:

$$H^i(G, H^j(\mathscr{F}_\Gamma(T))) \Rightarrow H^{i+j}(G, \mathscr{F}_\Gamma(T)).$$

Since $\mathscr{F}_\Gamma(-)$ is an exact functor, it follows that the initial term of the spectral sequence is isomorphic to $H^i(G, \mathscr{F}_\Gamma(H^j(T)))$ for every i and j, and therefore is a finitely generated Λ-module by we have proven above. Moreover, $H^n(G, \mathscr{F}_\Gamma(T))$ has a finite filtration consisting of subquotients of the $H^i(G, H^j(\mathscr{F}_\Gamma(T)))$ with $i + j = n$. Hence, $H^n(G, \mathscr{F}_\Gamma(T))$ is also a finitely generated Λ-module. □

Remark 3.2.5. In the case that Γ is abelian, Theorem 3.2.4 is a corollary of [Nek, Proposition 4.2.3], as $R[\![U]\!]$ is itself a commutative complete local Noetherian ring with finite residue field. In the case that T is a complex of R-projectives, it is a corollary of [FK, Proposition 1.6.5(2)].

3.3 Duals of induced modules

In this subsection, we describe the pairings and the resulting cup products that will be used in proving a duality theorem for the cohomology of induced modules. Key to our discussion is the following lemma, which we record for convenience. We let Ω and Σ denote auxiliary Iwasawa algebras: e.g., $\Omega = R[\![\Phi]\!]$ for some compact p-adic Lie group Φ.

Lemma 3.3.1. *Suppose that M is a bounded below complex of objects of $\mathcal{C}_{\Omega-\Lambda,G}$, let N be a bounded below complex of objects in $\mathcal{C}_{\Lambda-\Sigma,G}$, and let L be a bounded below complex of objects in $\mathcal{C}_{\Omega-\Sigma,G}$. Then any map*

$$\phi\colon M \hat{\otimes}_\Lambda N \longrightarrow L$$

of complexes in $\mathcal{C}_{\Omega-\Sigma,G}$ gives rise to a cup product morphism

$$\cup\colon C(G,M) \otimes_\Lambda C(G,N) \longrightarrow C(G,L)$$

of complexes of $\Omega \otimes_\Lambda \Sigma^\circ$-modules.

Proof. Briefly, the point is that to give such a map ϕ is equivalent to producing a collection of continuous, Λ-balanced, G-equivariant, left Ω-linear, and right Σ-linear pairings

$$\langle -,- \rangle_{mn}\colon M^m \times N^n \longrightarrow L^{m+n}$$

that are compatible with coboundaries in the sense of [Lim2, Section 2.3]. The cup product $\cup$ then arises from the induced cup products

$$\cup_{mn}^{ij}\colon C^i(G,M^m) \otimes_\Lambda C^j(G,N^n) \longrightarrow C^{i+j}(G,L^{m+n})$$

by combining them to a map of the total complexes with appropriate signs as in [Nek, (3.4.5.2)]: $\cup = ((-1)^{in}\cup_{mn}^{ij})$. □

Let us fix some notation. Let $\iota\colon \Lambda \to \Lambda$ denote the unique continuous R-algebra homomophism that satisfies $\iota(\gamma) = \gamma^{-1}$ for all $\gamma \in \Gamma$. If A is a $\Lambda[G]$-module, we let A^ι denote the $\Lambda^\circ[G]$-module that is A as an $R[G]$-module but on which a general $\lambda \in \Lambda$ now acts by left multiplication by $\iota(\lambda)$. We extend this to complexes in the obvious fashion. Of particular interest to us is $\mathscr{F}_\Gamma(T)^\iota$ for a complex T of objects in $\mathcal{C}_{R,G}$, in which case we have a canonical isomorphism

$$\mathscr{F}_\Gamma(T)^\iota \cong {}_\chi\Lambda \hat{\otimes}_R T,$$

of complexes in $\mathcal{C}_{\Lambda^\circ,G}$, since ι induces an isomorphism of $\Lambda[G]$-modules ${}_\chi\Lambda \xrightarrow{\sim} \Lambda^\iota_\chi$.

Lemma 3.3.2. *For objects M and N in $\mathcal{C}_{R,G}$, the pairing*

$$\begin{aligned} \langle -,- \rangle\colon \mathscr{F}_\Gamma(M) \times \mathscr{F}_\Gamma(N)^\iota &\longrightarrow \Lambda \hat{\otimes}_R (M \hat{\otimes}_R N) \\ (\lambda \otimes m, \mu \otimes n) &\mapsto \iota(\lambda)\mu \otimes m \otimes n \end{aligned}$$

is continuous, R-balanced, G-equivariant with respect to the trivial action on Λ in the target, and is left and right Λ-linear. Moreover, if M and N are any complexes of objects in $\mathcal{C}_{R,G}$, we have a morphism

$$\phi\colon \mathscr{F}_\Gamma(M)\,\hat{\otimes}_R\,\mathscr{F}_\Gamma(N)^\iota \longrightarrow \Lambda\,\hat{\otimes}_R\,(M\,\hat{\otimes}_R\,N)$$

of complexes in $\mathcal{C}_{\Lambda-\Lambda,G}$.

Proof. We begin by remarking that by Lemma 2.4.1, it makes sense to define the pairing on pairs of tensors, identified with their images in the completed tensor product. The pairing is clearly R-balanced, and the reader can check that it preserves the $\Lambda[G]$-Λ-actions. The continuity of the pairing reduces immediately to the continuity of

$$\Lambda \times \Lambda \longrightarrow \Lambda, \qquad (\lambda, \mu) \mapsto \iota(\lambda)\mu,$$

which in turn follows from the continuity of ι and the fact that Λ is a topological ring. The second statement follows easily. □

We next apply these pairings to study a duality between induced modules. As described in the introduction, the dualizing complex for R is an object ω_R of $\mathbf{D}^b_{R-\mathrm{ft}}(\mathrm{Mod}_R)$ with the property that for every object M of $\mathbf{D}(\mathrm{Mod}_R^{R-\mathrm{ft}})$, the object $\mathbf{R}\mathrm{Hom}_R(M, \omega_R)$ lies in $\mathbf{D}_{R-\mathrm{ft}}(\mathrm{Mod}_R)$ and the canonical morphism

$$M \longrightarrow \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, \omega_R), \omega_R)$$

is an isomorphism in $\mathbf{D}(\mathrm{Mod}_R)$. A dualizing complex exists for R and is unique up to translation and isomorphism in $\mathbf{D}^b_{R-\mathrm{ft}}(\mathrm{Mod}_R)$ (see [Har, Ch. V]). We choose a bounded complex J_R of injective R-modules which represents a choice of the dualizing complex in $\mathbf{D}(\mathrm{Mod}_R)$. (See [Nek, Section 0.4], or the proof of Lemma 4.3.1 below, for the construction of such a complex.)

Let T be a bounded complex of objects in $\mathcal{C}_{R,G}^{R-\mathrm{ft}}$. Then $\mathrm{Hom}_R(T, J_R)$ is a bounded complex of "admissible" $R[G]$-modules with cohomology groups that are finitely generated over R (see [Nek, (4.3.2)]). By [Nek, Proposition 3.3.9], there is a subcomplex T^* of $\mathrm{Hom}_R(T, J_R)$ which is a complex of objects in $\mathcal{C}_{R,G}^{R-\mathrm{ft}}$ (giving said objects the $\mathfrak{m}$-adic topology) and is quasi-isomorphic to $\mathrm{Hom}_R(T, J_R)$ via the inclusion map.

We have a composite morphism

$$\pi\colon T \otimes_R T^* \longrightarrow T \otimes_R \mathrm{Hom}_R(T, J_R) \longrightarrow J_R$$

of complexes of $R[G]$-modules, the first morphism being induced by the inclusion and the second being the usual evaluation map. By Lemma 3.3.2, this in turn induces a composite map

$$\overline{\pi}\colon \mathscr{F}_\Gamma(T)\,\hat{\otimes}_R\,\mathscr{F}_\Gamma(T^*)^\iota \xrightarrow{\phi} \Lambda\otimes_R T\otimes_R T^* \xrightarrow{\mathrm{id}\otimes\pi} \Lambda\otimes_R J_R$$

of complexes of $(\Lambda\otimes_R\Lambda^\circ)[G]$-modules. Twisting $\overline{\pi}$ by the identity map on an auxilliary complex of objects of $\mathcal{C}^{\Lambda^\circ-\mathrm{ft}}_{\Omega-\Lambda}$, Lemma 3.3.1 now provides the cup product morphisms of the following lemma, which will be used in the next section.

Lemma 3.3.3. *Let A be a complex of objects in $\mathcal{C}^{\Lambda^\circ-\mathrm{ft}}_{\Omega-\Lambda}$. The map $\overline{\pi}$ defined above induces a map*

$$\overline{\pi}_A\colon A\otimes_\Lambda \mathscr{F}_\Gamma(T)\,\hat{\otimes}_R\,\mathscr{F}_\Gamma(T^*) \longrightarrow A\otimes_R J_R$$

and, in turn, a well-defined cup product

$$C(G, A\otimes_\Lambda \mathscr{F}_\Gamma(T))\otimes_R C(G,\mathscr{F}_\Gamma(T^*)(1)) \longrightarrow C(G, A\otimes_R J_R(1)),$$

which is a map of complexes of $\Omega\otimes_R\Lambda^\circ$-modules.

Note that, by definition, the adjoint maps (defined as in [Lim2, Lemma 1.2])

$$\mathrm{adj}(\pi)\colon T^* \longrightarrow \mathrm{Hom}_R(T, J_R) \quad\text{and}\quad \mathrm{adj}'(\pi)\colon T\longrightarrow \mathrm{Hom}_R(T^*, J_R)$$

of π are quasi-isomorphisms, the first being simply the inclusion of complexes already defined. Since the terms of J_R are injective R-modules, it follows from Proposition 2.2.1 that the derived adjoint maps

$$\mathbf{adj}(\pi)\colon T^* \longrightarrow \mathbf{R}\mathrm{Hom}_R(T,\omega_R) \quad\text{and}\quad \mathbf{adj}'(\pi)\colon T\longrightarrow \mathbf{R}\mathrm{Hom}_R(T^*,\omega_R)$$

are isomorphisms in $\mathbf{D}^b_{R-\mathrm{ft}}(\mathrm{Mod}_{R[G]})$, hence in $\mathbf{D}^b_{R-\mathrm{ft}}(\mathscr{I}_{R,G})$, considering the restriction of $\mathbf{R}\mathrm{Hom}_R(-,-)$ to a bifunctor

$$\mathbf{D}^+(\mathcal{C}^{R-\mathrm{ft}}_{R,G})\times\mathbf{D}^-(\mathrm{Mod}_R)\longrightarrow \mathbf{D}^+(\mathscr{I}_{R,G}),$$

(see [Nek, (3.5.9)]).

Though we shall not use it later, we feel it important to note that the derived adjoint maps of $\overline{\pi}$ are also isomorphisms. While this is easy enough to prove in the derived category $\mathbf{D}(\mathrm{Mod}_{\Lambda^\circ[G]})$ of abstract modules (in the case of $\mathbf{adj}(\bar{\pi})$), we are interested in G-cohomology groups, so we want such an isomorphism in $\mathbf{D}(\mathscr{I}_{\Lambda^\circ,G})$.

Theorem 3.3.4. *For any bounded complexes T and T^* of objects in $\mathcal{C}_{R,G}^{R-\mathrm{ft}}$ such that T^* sits quasi-isomorphically as a subcomplex of the dual $\mathrm{Hom}_R(T, J_R)$, the derived adjoint maps*

$$\mathbf{adj}(\overline{\pi})\colon \mathscr{F}_\Gamma(T^*)^\iota \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda,\mathrm{cts}}(\mathscr{F}_\Gamma(T), \Lambda \otimes_R^{\mathbf{L}} \omega_R)$$
$$\big(\textit{resp.}, \ \mathbf{adj}'(\overline{\pi})\colon \mathscr{F}_\Gamma(T) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ,\mathrm{cts}}(\mathscr{F}_\Gamma(T^*)^\iota, \Lambda \otimes_R^{\mathbf{L}} \omega_R)\big)$$

are isomorphisms in $\mathbf{D}(\mathscr{I}_{\Lambda^\circ,G})$ (resp., $\mathbf{D}(\mathscr{I}_{\Lambda,G})$). Moreover, $\mathbf{R}\mathrm{Hom}_{\Lambda,\mathrm{cts}}(\mathscr{F}_\Gamma(T), \Lambda \otimes_R^{\mathbf{L}} \omega_R)$ (resp., $\mathbf{R}\mathrm{Hom}_{\Lambda^\circ,\mathrm{cts}}(\mathscr{F}_\Gamma(T^*)^\iota, \Lambda \otimes_R^{\mathbf{L}} \omega_R)$) can be represented by $\mathrm{Hom}_\Lambda(\mathscr{F}_\Gamma(T), \Lambda \otimes_R J_R)$ (resp., $\mathrm{Hom}_{\Lambda^\circ}(\mathscr{F}_\Gamma(T^*)^\iota, \Lambda \otimes_R J_R)$).

Proof. Let $X = \mathbf{R}\mathrm{Hom}_{\Lambda,\mathrm{cts}}(\mathscr{F}_\Gamma(T), \Lambda \otimes_R^{\mathbf{L}} \omega_R)$. Since J_R is a complex of R-injectives and ${}_\chi\Lambda$ is flat over R, Lemma 2.5.9 implies that the functor F given by

$$\mathrm{Hom}_{\Lambda,\mathrm{cts}}(\Lambda_\chi \hat{\otimes}_R -, \Lambda \otimes_R J_R)\colon \mathrm{Ch}^-(\mathcal{C}_{R,G})^\circ \longrightarrow \mathrm{Ch}^-(\mathscr{I}_{\Lambda^\circ,G})$$

is exact. In particular, if $P \xrightarrow{\sim} T$ is a quasi-isomorphism with P a bounded above complex of projective objects in $\mathcal{C}_{R,G}$, then $F(T) \to F(P)$ is a quasi-isomorphism. As $\Lambda_\chi \hat{\otimes}_R P$ is a complex of objects in $\mathcal{C}_{\Lambda,G}$ that are projective in $\mathcal{C}_\Lambda$, we then have that $F(T)$ represents X by Proposition 2.5.8.

By Lemmas 2.4.2(b) and 2.5.7, the functor F agrees with $\mathrm{Hom}_\Lambda(\Lambda_\chi \otimes_R -, \Lambda \otimes_R J_R)$ on $\mathrm{Ch}^-(\mathcal{C}_{R,G}^{R-\mathrm{ft}})^\circ$. Therefore, the commutative diagram in Proposition 2.5.8 tells us that X is represented by $\mathrm{Hom}_\Lambda(\mathscr{F}_\Gamma(T), \Lambda \otimes_R J_R)$ in $\mathbf{D}(\mathscr{I}_{\Lambda^\circ,G})$. Finally, since the map

$$T^* \longrightarrow \mathrm{Hom}_R(T, J_R)$$

is a quasi-isomorphism, the adjoint map of complexes

$${}_\chi\Lambda \otimes_R T^* \longrightarrow \mathrm{Hom}_\Lambda(\Lambda_\chi \otimes_R T, \Lambda \otimes_R J_R)$$

is a quasi-isomorphism of complexes of $\Lambda^\circ[G]$-modules by Lemma 2.2.3. Therefore, we have the result for $\mathbf{adj}(\overline{\pi})$, and the proof for $\mathbf{adj}'(\overline{\pi})$ is analogous. □

4 Duality over p-adic Lie extensions

We now turn to arithmetic. Here, we fix the notation that we shall use throughout this section. To start, let p be a prime. We let F be a global field of characteristic not equal to p. Let S be a finite set of primes of F that, in the case that F is a number field, contains

all primes above p and all real places. Let $S_{\mathbb{R}}$ denote the set of real places of F. Let $G_{F,S}$ denote the Galois group of the maximal unramified outside S extension of F. For a place v, let F_v denote the completion of F at v, and let G_v denote a decomposition group in the absolute Galois group G_F of F.

We fix a p-adic Lie extension F_∞ of F that is unramified outside S, and we let Γ denote its Galois group. We let R denote a complete commutative local Noetherian ring with finite residue field of characteristic p, and we set $\Lambda = R[\![\Gamma]\!]$.

4.1 Iwasawa cohomology

We recall the following facts, all of which can all be found in [NSW, Chapters VII-VIII]. The $G_{F,S}$-cohomology groups of a finite $G_{F,S}$-module of p-power order are all finite. If p is odd or F has no real places, then $G_{F,S}$ has p-cohomological dimension at most 2, and so is p-cohomologically finite in the sense of Section 3.2. If v is a nonarchimedean place that does not lie above the characteristic of F, then G_v has p-cohomological dimension equal to 2, and the G_v-cohomology groups of a finite G_v-module of p-power order are finite as well. Of course, the G_v-cohomology groups of a finite module are also finite for archimedean v, since G_v is of order dividing 2 for such places.

Recall (e.g., from [Nek, (5.7.2)]) that, for any profinite ring Ω, the (Tate) compactly supported $G_{F,S}$-cochain complex of a bounded complex of objects M in $\mathcal{C}_{\Omega,G_{F,S}}$ is defined as

$$C_{(c)}(G_{F,S}, M) = \mathrm{Cone}\left(C(G_{F,S}, M) \longrightarrow \bigoplus_{v \in S - S_{\mathbb{R}}} C(G_v, M) \oplus \bigoplus_{v \in S_{\mathbb{R}}} \hat{C}(G_v, M)\right)[-1],$$

where $\hat{C}(G_v, M)$ is defined as in [Lim2, Section 2.4] and denotes the standard complete complex of Tate G_v-cochains for M.

We remark that, in Nekovář's notation, $C_{(c)}(G_{F,S}, M)$ is denoted $\hat{C}_c(G_{F,S}, M)$. We use the notation in [FK], which uses the parentheses to distinguish the latter group from the compactly supported cochains $C_c(G_{F,S}, M)$, for which one uses the usual cohomology groups at real places. To shorten notation in what follows, for $v \in S_{\mathbb{R}}$, we will abuse notation and use $C(G_v, M)$, $H^i(G_v, M)$, and $\mathbf{R}\Gamma(G_v, M)$ to denote the Tate cochains $\hat{C}(G_v, M)$, its cohomology groups, and its derived object. We denote the derived object corresponding to $C_{(c)}(G_{F,S}, M)$ by $\mathbf{R}\Gamma_{(c)}(G_{F,S}, M)$ and its ith cohomology group by

$H^i_{(c)}(G_{F,S}, M)$. By definition, we have an exact triangle[2]

$$\mathbf{R}\Gamma_{(c)}(G_{F,S}, M) \longrightarrow \mathbf{R}\Gamma(G_{F,S}, M) \longrightarrow \bigoplus_{v \in S} \mathbf{R}\Gamma(G_v, M)$$

in $\mathbf{D}(\mathrm{Mod}_\Omega)$. By our above remarks, if F has no real places or p is odd, then the triangle lies in $\mathbf{D}^b(\mathrm{Mod}_\Omega)$. The following is now an immediate consequence of Corollary 3.1.6 and Remark 3.1.4, with the first horizontal isomorphism being defined by the naturality of the cone.

Proposition 4.1.1. *Let Γ' be quotient of Γ by a closed subgroup, and set $\Lambda' = R[\![\Gamma']\!]$. Let T be a bounded complex of objects in $\mathcal{C}_{R,G_{F,S}}$. Then we have an isomorphism of exact triangles*

$$\begin{array}{ccc}
\Lambda' \otimes^{\mathbf{L}}_{\Lambda} \mathbf{R}\Gamma_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T)) & \xrightarrow{\sim} & \mathbf{R}\Gamma_{(c)}(G_{F,S}, \mathscr{F}_{\Gamma'}(T)) \\
\downarrow & & \downarrow \\
\Lambda' \otimes^{\mathbf{L}}_{\Lambda} \mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_\Gamma(T)) & \xrightarrow{\sim} & \mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_{\Gamma'}(T)) \\
\downarrow & & \downarrow \\
\Lambda' \otimes^{\mathbf{L}}_{\Lambda} \bigoplus_{v \in S} \mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T)) & \xrightarrow{\sim} & \bigoplus_{v \in S} \mathbf{R}\Gamma(G_v, \mathscr{F}_{\Gamma'}(T))
\end{array}$$

in $\mathbf{D}(\mathrm{Mod}_{\Lambda'})$.

In the next proposition, we note that if T is a bounded complex of objects of $\mathcal{C}^{R-\mathrm{ft}}_{R,G_{F,S}}$, then the global, local, and compactly supported cohomology groups of $\mathscr{F}_\Gamma(T)$ are finitely generated Λ-modules. Note that any object of $\mathcal{C}^{R-\mathrm{ft}}_{R,G_{F,S}}$ may be considered to be an object of $\mathcal{C}^{R-\mathrm{ft}}_{R,G_v}$ via the canonical map $G_v \to G_{F,S}$.

Proposition 4.1.2. *Suppose that T is a bounded complex of objects of $\mathcal{C}^{R-\mathrm{ft}}_{R,G_{F,S}}$. Then $\mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_\Gamma(T))$ and $\mathbf{R}\Gamma_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T))$ are objects of $\mathbf{D}^+_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$ and $\mathbf{D}^-_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$, respectively . Similarly, if T is a bounded complex of objects in $\mathcal{C}^{R-\mathrm{ft}}_{R,G_v}$ for some place $v \in S$, then $\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T))$ lies in $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$ (and $\mathbf{D}^b_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$ if v is nonarchimedean). If F has no real places or p is odd, then all of these derived objects lie in $\mathbf{D}^b_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$.*

Proof. If $v \in S$ is nonarchimedean, then G_v is p-cohomologically finite by our above remarks, and hence Theorem 3.2.4 implies that $\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T))$ lies in $\mathbf{D}^b_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$. In the case that p is odd or F has no real places, Theorem 3.2.4 again implies immediately that $\mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_\Gamma(T))$ lies in $\mathbf{D}^b_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$. In this case, it follows from the above-mentioned exact triangle that $\mathbf{R}\Gamma_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T))$ is in $\mathbf{D}^b_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$.

[2] We write an exact triangle $A \to B \to C \to A[1]$ more compactly as $A \to B \to C$ throughout.

In the general case, we remark that $H^i_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T))$ for a module T vanishes in dimension greater than 3 [Nek, Lemma 5.7.3], and, moreover, is finite on finite $G_{F,S}$-modules (which follows from its definition as a cone and the same fact for global and local cohomology). The proof of Theorem 3.2.4 carries over exactly to the case of compactly supported cohomology, notwithstanding the fact that $H^i_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T))$ for an object T may in general be nonzero for negative i, though that precludes boundedness. Therefore, we obtain that $\mathbf{R}\Gamma_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T))$ is in $\mathbf{D}^-_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$. For real v, any $i \in \mathbb{Z}$ and T concentrated in degree 0, the Tate cohomology group $H^i(G_v, \mathscr{F}_\Gamma(T))$ is a 2-torsion subquotient of $\mathscr{F}_\Gamma(T)$, as G_v has order 2. Hence, it is finitely generated over Λ, as $\mathscr{F}_\Gamma(T)$ is. The argument given in the proof of Theorem 3.2.4 then allows one to pass to the case of a bounded complex, and therefore $\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T))$ lies in $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$ for real (and hence any) $v \in S$. It then follows from the exact triangle that $\mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_\Gamma(T))$ sits in $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$, and hence in $\mathbf{D}^+_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$, as the $G_{F,S}$-cohomology groups of a module vanish in negative dimensions. □

Remark 4.1.3. We may also consider the setting in which we take F itself to be a nonarchimedean local field of characteristic not equal to p. We then let G_F be its absolute Galois group, let Γ denote the Galois group of a p-adic Lie extension of F, and as before, set $\Lambda = R[\![\Gamma]\!]$. In this case, we obtain immediately from Theorem 3.2.4 that the cohomology groups $H^j(G_F, \mathscr{F}_\Gamma(T))$ are finitely generated Λ-modules for all j and any object T of $\mathcal{C}^{R-\mathrm{ft}}_{R,G_F}$, and consequently, that $\mathbf{R}\Gamma(G_F, \mathscr{F}_\Gamma(T))$ is an object of $\mathbf{D}^b_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$ for any bounded complex T of objects in $\mathcal{C}^{R-\mathrm{ft}}_{R,G_F}$.

4.2 Duality over local fields

In this section, we will prove a version of Tate duality with the dualizing complex ω_R replacing $\mathbb{Q}_p/\mathbb{Z}_p$. Fix a nonarchimedean prime $v \in S$. We continue to suppose that Γ is a compact p-adic Lie group that is a quotient of $G_{F,S}$ defining an extension F_∞, though as we remark later, we could just as well assume it to be a quotient of G_v. Let T be a bounded complex of objects in $\mathcal{C}^{R-\mathrm{ft}}_{R,G_v}$. As before, we let J_R denote a complex of injective R-modules that represents ω_R, and we let T^* be a quasi-isomorphic subcomplex of $\mathrm{Hom}_R(T, J_R)$ consisting of objects of $\mathcal{C}^{R-\mathrm{ft}}_{R,G_v}$.

For a double complex X (or its total complex) and $n \in \mathbb{Z}$, we let $\tau^{II}_{\geq n}(X)$ (resp., $\tau^{II}_{\leq n}(X)$) be the total complex of the quotient complex of X with jth column equal to

$$\tau_{\geq n}(X^{\bullet j}) = [\cdots \to 0 \to 0 \to \operatorname{coker} d^n_{X^{\bullet j}} \to X^{n+1,j} \to X^{n+2,j} \to \cdots]$$
$$\big(\text{resp., } \tau_{\leq n}(X^{\bullet j}) = [\cdots \to X^{n-2,j} \to X^{n-1,j} \to \ker d^n_{X^{\bullet j}} \to 0 \to 0 \to \cdots]\big)$$

The cup product of Lemma 3.3.3 then yields a map

$$C(G_v, \mathscr{F}_\Gamma(T)) \otimes_R C(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)) \longrightarrow \tau^{II}_{\geq 2} C(G_v, \Lambda \otimes_R J_R(1))$$

of $\Lambda \otimes_R \Lambda^\circ$-modules.

Taking adjoints, we have the following maps of complexes of Λ-modules and Λ°-modules, respectively:

$$C(G_v, \mathscr{F}_\Gamma(T)) \longrightarrow \mathrm{Hom}_{\Lambda^\circ}\Big(C(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \tau^{II}_{\geq 2} C(G_v, \Lambda \otimes_R J_R(1))\Big),$$
$$C(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)) \longrightarrow \mathrm{Hom}_{\Lambda}\Big(C(G_v, \mathscr{F}_\Gamma(T)), \tau^{II}_{\geq 2} C(G_v, \Lambda \otimes_R J_R(1))\Big).$$

From now on, we use $M^\vee$ to denote the Pontryagin dual of a locally compact abelian group M. We have the following analogue of [Nek, (5.2.1)].

Lemma 4.2.1. *Let A be a bounded complex in $\mathcal{C}^{\Lambda^\circ-\mathrm{ft}}_{\Lambda-\Lambda}$ with terms that are flat as R-modules. Then there exists a quasi-isomorphism*

$$q_A \colon (A \otimes_R J_R)[-2] \longrightarrow \tau^{II}_{\geq 2} C(G_v, A \otimes_R J_R(1)),$$

natural in A, of chain complexes of $\Lambda \otimes_R \Lambda^\circ$-modules.

Proof. We first note that if M is any finitely generated R-module with a trivial action of G_v, then Tate local duality as in [Lim2, Theorem 3.1.2] yields isomorphisms

$$H^2(G_v, A \otimes_R M(1)) \cong H^0(G_v, (A \otimes_R M)^\vee)^\vee \cong A \otimes_R M,$$

of $\Lambda \otimes_R \Lambda^\circ$-modules. Now suppose that M is any (ind-admissible) $R[G_v]$-module with trivial G_v-action, and write $M = \bigcup M_\alpha$, where the M_α are finitely generated R-submodules of M. Since the terms of A are R-flat, $A^i \otimes_R M$ is the union of the $A^i \otimes_R M_\alpha$ for any $i \in \mathbb{Z}$, so

$$H^2(G_v, A \otimes_R M(1)) \cong \varinjlim H^2(G_v, A \otimes_R M_\alpha(1)) \cong \varinjlim A \otimes_R M_\alpha \cong A \otimes_R M.$$

As G_v is of p-cohomological dimension 2, we therefore have a quasi-isomorphism

$$A \otimes_R M[-2] \longrightarrow \tau^{II}_{\geq 2} C(G_v, A \otimes_R M(1)),$$

of complexes of $\Lambda \otimes_R \Lambda^\circ$-modules. The case of a bounded complex M of ind-admissible R-modules, e.g. $M = J_R$, then follows easily. Naturality is immediate from the construction. $\square$

Combining Lemma 4.2.1 for $A = \Lambda$ with the morphisms constructed above and passing to the derived category, we obtain morphisms as in the following theorem.

Theorem 4.2.2. *Let T be a bounded complex in $\mathcal{C}^{R-\mathrm{ft}}_{R,G_v}$. Then the morphisms*

$$\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-2]$$
$$\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T)), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-2]$$

in $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$ and $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_{\Lambda^\circ})$, respectively, are isomorphisms.

Over the course of the next two subsections, we will show that the first of the morphisms of Theorem 4.2.2 is an isomorphism. The proof that the second is an isomorphism is completely analogous. By an argument similar to that in [Nek, Lemma 5.2.5], it suffices to prove the theorem for a particular translate of the dualizing complex. In view of this, we will assume that our choice of the dualizing complex satisfies [Nek, (2.5)(i)] throughout the next two subsections.

Remark 4.2.3. In the setting that F is a nonarchimedean local field of characteristic not equal to p, that F_∞ is any p-adic Lie extension of F, and that T is a bounded complex of objects in $\mathcal{C}^{R-\mathrm{ft}}_{R,G_F}$, the argument we are about to describe also carries over as in Remark 4.1.3 to prove the direct analogue of Theorem 4.2.2 with G_v replaced by G_F.

4.3 Change of rings

Recall that R is a complete commutative local Noetherian ring with maximal ideal $\mathfrak{m}$ and finite residue field k. In this subsection, we shall show that it suffices to prove Theorem 4.2.2 for any complete commutative local Noetherian ring S that has R as a quotient. That is, we suppose given a surjection $\psi\colon S \to R$, which necessarily induces an isomorphism on residue fields. We use $\mathfrak{m}_S$ to denote the maximal ideal of S, and we set

$$d = \dim_k \mathfrak{m}_S/\mathfrak{m}_S^2 - \dim_k \mathfrak{m}/\mathfrak{m}^2.$$

We note that if A is any complex of S-modules, then $\mathrm{Hom}_S(R, A)$ is isomorphic to the subcomplex of A with terms the torsion submodules under $\ker\psi$. In particular, if B is a complex of R-modules, then the resulting inclusion map induces an isomorphism

$$\mathrm{Hom}_R(B, \mathrm{Hom}_S(R, A)) \xrightarrow{\sim} \mathrm{Hom}_S(B, A),$$

as is easy enough to see by the usual adjointness principle. In particular, we have

$$\mathrm{Hom}_R(B, R^\vee) \xrightarrow{\sim} \mathrm{Hom}_S(B, S^\vee).$$

With this comparison of Matlis duals with respect to R and S in hand, we now compare Grothendieck duals with respect to these rings.

Lemma 4.3.1. *Let T be a bounded complex in $\mathcal{C}^{R-\mathrm{ft}}_{R,G_v}$, which we view also as a complex in $\mathcal{C}^{S-\mathrm{ft}}_{S,G_v}$ via ψ. Then ψ induces a natural map $J_R \to J_S[d]$ that in turn induces an isomorphism*

$$\mathrm{Hom}_R(T, J_R) \xrightarrow{\sim} \mathrm{Hom}_S(T, J_S)[d]$$

of complexes in $\mathscr{I}_{S,G_v}$.

Proof. Choose $\bar{x}_1, \ldots, \bar{x}_r \in \mathfrak{m}$ with images forming a k-basis of $\mathfrak{m}/\mathfrak{m}^2$. Recall (cf. [Nek, (0.4)]) that J_R may be taken as the Matlis dual of the complex $C_R[r]$, where C_R is defined to be the complex

$$[R \to \bigoplus_i R_{\bar{x}_i} \to \bigoplus_{i<j} R_{\bar{x}_i\bar{x}_j} \to \cdots \to R_{\bar{x}_1\cdots\bar{x}_r}]$$

in degrees $[0, r]$, with the usual Čech differentials, which is to say that

$$J_R = \mathrm{Hom}_R(C_R[r], R^\vee).$$

We lift $\bar{x}_1, \ldots, \bar{x}_r$ to elements $x_1, \ldots, x_r$ of $\mathfrak{m}_S$ and extend to a sequence $x_1, \ldots, x_s$ such that the images of $x_1, \ldots, x_s$ form a basis of $\mathfrak{m}_S/\mathfrak{m}_S^2$ and the elements $x_{r+1}, \ldots, x_s$ map trivially to $\mathfrak{m}/\mathfrak{m}^2$. Modifying each x_i with $i > r$ by an element of $\mathfrak{m}_S^2$ if necessary, we may choose these elements so that they in fact map trivially to $\mathfrak{m}$. We use this sequence to define C_S, and we have $J_S = \mathrm{Hom}_S(C_S[s], S^\vee)$. The natural map $C_S \to C_R$ factors through an isomorphism $R \otimes_S C_S \xrightarrow{\sim} C_R$. This in turn induces an isomorphism

$$J_R \xrightarrow{\sim} \mathrm{Hom}_S(R, J_S)[d],$$

which is given by the following composition of isomorphisms

$$\begin{aligned}\mathrm{Hom}_R(C_R[r], R^\vee) &\xrightarrow{\sim} \mathrm{Hom}_S(C_R[r], S^\vee) \xrightarrow{\sim} \mathrm{Hom}_S(R \otimes_S C_S[r], S^\vee) \\ &\xrightarrow{\sim} \mathrm{Hom}_S(R, \mathrm{Hom}_S(C_S[r], S^\vee)) \xrightarrow{\sim} \mathrm{Hom}_S(R, \mathrm{Hom}_S(C_S[s], S^\vee))[d],\end{aligned}$$

where the latter natural map is defined as in [Nek, (1.2.15)] (noting [Nek, (1.2.5)]). The result then follows from the isomorphisms

$$\mathrm{Hom}_R(T, J_R) \xrightarrow{\sim} \mathrm{Hom}_R(T, \mathrm{Hom}_S(R, J_S))[d] \xrightarrow{\sim} \mathrm{Hom}_S(T, J_S)[d].$$

□

We now proceed to the reductive step.

Proposition 4.3.2. *Let T be a bounded complex in $\mathcal{C}^{R-\mathrm{ft}}_{R,G_v}$. Setting $\Omega = S[\![\Gamma]\!]$, there is a commutative diagram of natural morphisms*

$$\begin{tikzcd}
\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T)) \arrow[r] \arrow[d, "\wr"] & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-2] \arrow[d, "\wr"] \\
\mathbf{R}\Gamma(G_v, \Omega^\iota \otimes_S T) \arrow[r] & \mathbf{R}\mathrm{Hom}_{\Omega^\circ}\Big(\mathbf{R}\Gamma(G_v, \Omega \otimes_S T^*(1)), \Omega \otimes^{\mathbf{L}}_S \omega_S[d]\Big)[-2]
\end{tikzcd}$$

in which the horizontal morphisms are as in Theorem 4.2.2 and the vertical arrows are isomorphisms in $\mathbf{D}(\mathrm{Mod}_\Omega)$. In particular, Theorem 4.2.2 holds for R if it holds for S.

Proof. Note that we have

$$\Omega_\chi \otimes_S T \cong \Lambda_\chi \otimes_R T \quad (\text{resp., } {}_\chi\Omega \otimes_S T^* \cong {}_\chi\Lambda \otimes_R T^*)$$

so $\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T))$ (resp., $\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1))$) is the same complex for R and for S. By Lemma 4.3.1, and employing the S-flatness of Ω in applying Lemma 2.1.7, we have canonical isomorphisms

$$\Lambda \otimes_R J_R \xrightarrow{\sim} \Omega \otimes_S J_R \xrightarrow{\sim} \Omega \otimes_S \mathrm{Hom}_S(R, J_S)[d] \xrightarrow{\sim} \mathrm{Hom}_S(R, \Omega \otimes_S J_S)[d],$$

of complexes of $\Omega^\circ \otimes_S R \cong \Lambda^\circ$-modules, and the composition of the composite isomorphism with the natural inclusion

$$\mathrm{Hom}_S(R, \Omega \otimes_S J_S)[d] \longrightarrow \Omega \otimes_S J_S[d]$$

induces the righthand vertical map in the proposition. That the diagram commutes then follows directly from the definition of the cup product in Lemma 3.3.3.

Let us fix a quasi-isomorphism $\iota\colon \Omega \otimes_S J_S \to K$, where K is a bounded below complex of injective Ω°-modules. We check that the map

$$\iota_R\colon\ \mathrm{Hom}_S(R, \Omega \otimes_S J_S) \longrightarrow \mathrm{Hom}_S(R, K)$$

induced by ι is a quasi-isomorphism, which implies that $\mathrm{Hom}_S(R, \Omega \otimes_S J_S)$ represents $\mathbf{R}\mathrm{Hom}_S(R, \Omega \otimes^{\mathbf{L}}_S \omega_S)$ in $\mathbf{D}(\mathrm{Mod}_{\Lambda^\circ})$. Let $\varepsilon\colon P \to R$ be a resolution of R by a complex P of finitely generated free S-modules. We have a commutative diagram

$$\begin{tikzcd}
\Omega \otimes_S \mathrm{Hom}_S(R, J_S) \arrow[r, "\phi_R"] \arrow[d, "\mathrm{id}\otimes\varepsilon^*_J"] & \mathrm{Hom}_S(R, \Omega \otimes_S J_S) \arrow[r, "\iota_R"] \arrow[d, "\varepsilon^*_{\Omega\otimes J}"] & \mathrm{Hom}_S(R, K) \arrow[r, "\psi_R"] \arrow[d, "\varepsilon^*_K"] & \mathrm{Hom}_{\Omega^\circ}(\Omega \otimes_S R, K) \arrow[d, "(\mathrm{id}\otimes\varepsilon)^*_K"] \\
\Omega \otimes_S \mathrm{Hom}_S(P, J_S) \arrow[r, "\phi_P"] & \mathrm{Hom}_S(P, \Omega \otimes_S J_S) \arrow[r, "\iota_P"] & \mathrm{Hom}_S(P, K) \arrow[r, "\psi_P"] & \mathrm{Hom}_{\Omega^\circ}(\Omega \otimes_S P, K)
\end{tikzcd}$$

in which the vertical maps are all induced by ε. Moreover, note that the upper horizontal morphisms are of complexes of Λ°-modules and the others are of complexes of Ω°-modules. The maps ϕ_P and ϕ_R are isomorphisms, and the map $\mathrm{id}\otimes\varepsilon_J^*$ is a quasi-isomorphism by the S-injectivity of J_S and the S-flatness of Ω, so $\varepsilon^*_{\Omega\otimes J}$ is also a quasi-isomorphism. The maps ψ_R and ψ_P are isomorphisms by the usual adjointness of Hom and the tensor product, and the map $(\mathrm{id}\otimes\varepsilon)^*_K$ is a quasi-isomorphism by the Ω°-injectivity of K, so ε^*_K is a quasi-isomorphism as well. Finally, the map ι_P is a quasi-isomorphism by the S-projectivity of P, and it follows that ι_R is a quasi-isomorphism.

To finish the proof, we need only show that the morphism

$$\mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(X, \mathbf{R}\mathrm{Hom}_S(R, \Omega \otimes^{\mathbf{L}}_S \omega_S)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Omega^\circ}(X, \Omega \otimes^{\mathbf{L}}_S \omega_S)$$

is an isomorphism. It is easy to see that $\mathrm{Hom}_S(R, K)$ is a complex of injective Λ°-modules. Moreover, every Ω°-homomorphism from the complex X of Λ°-modules to K must factor through

$$\mathrm{Hom}_{\Omega^\circ}(\Lambda, K) \cong \mathrm{Hom}_{\Omega^\circ}(\Omega \otimes_S R, K) \cong \mathrm{Hom}_S(R, K)$$

of K, so the map

$$\mathrm{Hom}_{\Lambda^\circ}(X, \mathrm{Hom}_S(R, K)) \longrightarrow \mathrm{Hom}_{\Omega^\circ}(X, K)$$

induced by inclusion is an isomorphism, as desired. □

4.4 Duality over flat $\mathbb{Z}_p$-algebras

In this subsection, we will prove Theorem 4.2.2. As remarked in the proof of Proposition 3.0.1, every complete commutative Noetherian local ring with finite residue field is a quotient of a power series ring in finitely many variables over an unramified extension of $\mathbb{Z}_p$. Proposition 4.3.2 then allows us to reduce Theorem 4.2.2 to the case of such power series rings. As these rings are flat over $\mathbb{Z}_p$, it suffices to assume that R is $\mathbb{Z}_p$-flat, which we do throughout the remainder of this subsection.

We will require a bit more general of a derived adjoint map than the one we wish to prove is an isomorphism. The construction is found in the following easy lemma.

Lemma 4.4.1. *Let A be a bounded complex of R-flat objects in $\mathcal{C}^{\Lambda^\circ-\mathrm{ft}}_{\Lambda-\Lambda}$. Then we have a morphism*

$$\mathbf{R}\Gamma(G_v, A \otimes_\Lambda \mathscr{F}_\Gamma(T)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), A \otimes^{\mathbf{L}}_R \omega_R\Big)[-2]$$

in $\mathbf{D}(\mathrm{Mod}_\Lambda)$ that is natural in the complex A.

Proof. This is simply the derived adjoint of the composition of the cup product in Lemma 3.3.3 with the map to the truncation $\tau^{II}_{\geq 2}C(G_v, A \otimes_R J_R(1))$, which is quasi-isomorphic to $(A \otimes_R J_R)[-2]$. Naturality is immediate. $\square$

We will mostly be interested in complexes of length 2, so we introduce the following notation. Suppose that B is an object in $\mathcal{C}^{\Lambda^\circ-\mathrm{ft}}_{\Lambda-\Lambda}$ and that A is a subobject of B, which is to say that it is a closed $\Lambda \hat{\otimes}_R \Lambda^\circ$-submodule with the subspace topology. Then for any bounded complex T of objects in $\mathcal{C}^{R-\mathrm{ft}}_{R,G_v}$, we define a complex

$$\mathscr{F}_{B/A}(T) = [A \to B] \otimes_\Lambda \mathscr{F}_\Gamma(T),$$

of objects in $\mathcal{C}_{\Lambda,G_v}$, where A and B are in degree -1 and 0 respectively.

Let $I = I(\Gamma)$ denote the augmentation ideal of Λ. We then have the following lemmas. Recall for the first that we assume that R is $\mathbb{Z}_p$-flat in this subsection.

Lemma 4.4.2. *The following statements hold for any $n \geq 0$.*

(*a*) *The module I^n is a flat R-module.*

(*b*) *The module I^n/I^{n+1} is finitely generated and of finite projective dimension over R.*

Proof. (a) Suppose first that Γ is finite, and let $\mathcal{I}$ denote the augmentation ideal in $\mathbb{Z}_p[\Gamma]$. As $\mathbb{Z}_p[\Gamma]$ is finitely generated and free over $\mathbb{Z}_p$, so is $\mathcal{I}^n$. Since R is $\mathbb{Z}_p$-flat, the natural surjection $R \otimes_{\mathbb{Z}_p} \mathcal{I}^n \to I^n$ is an isomorphism, and therefore I^n is free over R.

In the general case, $\Gamma \cong \varprojlim \Gamma'$, where Γ' runs over the Galois groups of the finite Galois extensions of F in F_∞, and I^n is the inverse limit of the nth powers of the augmentation ideals $I(\Gamma')$ of the $R[\Gamma']$. As $I(\Gamma')^n \cong I^n \otimes_\Lambda R[\Gamma']$, an application of Lemma 2.4.2 yields an isomorphism

$$I^n \otimes_R B \xrightarrow{\sim} \varprojlim_{\Gamma'} (I(\Gamma')^n \otimes_R B)$$

for any ideal B of R. The left exactness of the inverse limit and the R-flatness of $I(\Gamma')^n$ then imply that the canonical map $I^n \otimes_R B \to I^n$ is an injection, proving the flatness of I^n.

(b) Let $\mathcal{I}$ denote the augmentation ideal in $\mathbb{Z}_p[\![\Gamma]\!]$. For each n, the composition

$$R \hat{\otimes}_{\mathbb{Z}_p} \mathcal{I}^n \longrightarrow R \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[\![\Gamma]\!] \xrightarrow{\sim} R[\![\Gamma]\!]$$

induces a surjection $R\hat{\otimes}_{\mathbb{Z}_p}\mathcal{I}^n \twoheadrightarrow I^n$ that fits into the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & R\hat{\otimes}_{\mathbb{Z}_p}\mathcal{I}^{n+1} \arrow[r] \arrow[d, two heads] & R\hat{\otimes}_{\mathbb{Z}_p}\mathcal{I}^{n} \arrow[r] \arrow[d, two heads] & R\hat{\otimes}_{\mathbb{Z}_p}(\mathcal{I}^n/\mathcal{I}^{n+1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^{n+1} \arrow[r] & I^n \arrow[r] & I^n/I^{n+1} \arrow[r] & 0.
\end{tikzcd}$$

Since the two vertical maps on the left are surjections, so is the one on the right. Note that $\mathcal{I}^n/\mathcal{I}^{n+1}$ is a quotient of nth tensor power of the maximal abelian pro-p quotient of Γ, a finitely generated $\mathbb{Z}_p$-module. Therefore, the existence of the above surjection implies that I^n/I^{n+1} is finitely generated over R. To see that the last assertion holds, we note that it follows from (a) that I^n/I^{n+1} has finite flat dimension. Since flat dimension coincides with projective dimension for every finitely generated module over a Noetherian ring (see [Wei, Proposition 4.1.5]), we have our assertion. □

Lemma 4.4.3. *Let T be a bounded complex of objects in $\mathcal{C}^{R-\mathrm{ft}}_{R,G_v}$. Then we have a quasi-isomorphism*

$$\mathrm{Cone}\Big(\mathscr{F}_{I^n/I^{n+1}}(T) \longrightarrow \mathscr{F}_{\Lambda/I^{n+1}}(T)\Big) \xrightarrow{\sim} \mathscr{F}_{\Lambda/I^n}(T)$$

of complexes in $\mathcal{C}^{\Lambda-\mathrm{ft}}_{\Lambda,G_v}$. Moreover, we have an exact triangle

$$I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R \longrightarrow \Lambda/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R \longrightarrow \Lambda/I^n \otimes^{\mathbf{L}}_R \omega_R$$

in $\mathbf{D}^b(\mathrm{Mod}_{\Lambda\otimes_R\Lambda^\circ})$.

Proof. Since the powers of I are R-flat by Lemma 4.4.2, it suffices in both cases to show that there is a quasi-isomorphism

$$\mathrm{Cone}\big([I^{n+1} \to I^n] \longrightarrow [I^{n+1} \to \Lambda]\big) \xrightarrow{\sim} [I^n \to \Lambda].$$

Note that the latter cone is precisely the complex

$$I^{n+1} \xrightarrow{f} I^{n+1} \oplus I^n \xrightarrow{g} \Lambda,$$

where $f(x) = (x, -x)$ and $g(x, y) = x + y$ for $x \in I^{n+1}$ and $y \in I^n$. One can now easily check that the diagram

$$\begin{tikzcd}
I^{n+1} \arrow[r] \arrow[d] & I^{n+1} \oplus I^n \arrow[r] \arrow[d, "\nu"'] & \Lambda \arrow[d, equal] \\
0 \arrow[r] & I^n \arrow[r] & \Lambda
\end{tikzcd}$$

commutes, where ν is given by $\nu(x, y) = x + y$ for $x \in I^{n+1}$ and $y \in I^n$, and the vertical maps induce isomorphisms on cohomology. □

We are now able to prove the following proposition, which is an important ingredient in the proof of Theorem 4.2.2.

Proposition 4.4.4. *Let T be a bounded complex of objects in $\mathcal{C}_{R,G_v}^{R-\mathrm{ft}}$. Then we have the following morphism of exact triangles*

$$\begin{tikzcd}
\mathbf{R}\Gamma(G_v, \mathscr{F}_{I^n/I^{n+1}}(T)) \arrow[r] \arrow[d] & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), I^n/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R\Big)[-2] \arrow[d] \\
\mathbf{R}\Gamma(G_v, \mathscr{F}_{\Lambda/I^{n+1}}(T)) \arrow[r] \arrow[d] & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R\Big)[-2] \arrow[d] \\
\mathbf{R}\Gamma(G_v, \mathscr{F}_{\Lambda/I^{n}}(T)) \arrow[r] & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda/I^{n} \otimes_R^{\mathbf{L}} \omega_R\Big)[-2]
\end{tikzcd}$$

in $\mathbf{D}(\mathrm{Mod}_\Lambda)$.

Proof. By Lemma 4.4.2(a), we see that $I^n/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R$ and $\Lambda/I^n \otimes_R^{\mathbf{L}} \omega_R$ are represented by $[I^{n+1} \to I^n] \otimes_R J_R$ and $[I^n \to \Lambda] \otimes_R J_R$ respectively. Therefore, the commutativity of the diagram in the proposition follows from the naturality in Lemma 4.4.1. By Lemma 4.4.3, both columns are exact triangles. □

We now describe the idea of the proof of Theorem 4.2.2. We shall first prove that the morphism

$$\mathbf{R}\Gamma(G_v, \mathscr{F}_{\Lambda/I^n}(T)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda/I^n \otimes_R^{\mathbf{L}} \omega_R\Big)[-2]$$

is an isomorphism for all n. Then, Theorem 4.2.2 will follow from this by a limit argument. To show that above morphism is an isomorphism, we will utilize Proposition 4.4.4. Note that if any two of the horizontal morphisms in Proposition 4.4.4 is a quasi-isomorphism, so is the third one. Therefore, by an inductive argument, we are reduced to showing that the following morphism

$$\mathbf{R}\Gamma(G_v, \mathscr{F}_{I^n/I^{n+1}}(T)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), I^n/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R\Big)[-2]$$

is an isomorphism for all $n \geq 0$.

Note that Γ acts trivially on I^n/I^{n+1}. Therefore, one may view I^n/I^{n+1} as a Λ°-module via the augmentation map $\Lambda \to R$. We now have the following lemma.

Lemma 4.4.5. *Let T be a bounded complex of objects in $\mathcal{C}_{R,G_v}^{R-\mathrm{ft}}$. Then we have the following isomorphisms*

$$\mathscr{F}_{I^n/I^{n+1}}(T) \xrightarrow{\sim} I^n/I^{n+1} \hat{\otimes}_{\Lambda}^{\mathbf{L}} \mathscr{F}_{\Gamma}(T) \xrightarrow{\sim} I^n/I^{n+1} \hat{\otimes}_{R}^{\mathbf{L}} T$$

in $\mathbf{D}^b(\mathcal{C}_{\Lambda,G_v})$. Therefore, it follows that we have an isomorphism

$$\mathbf{R}\Gamma(G_v, \mathscr{F}_{I^n/I^{n+1}}(T)) \xrightarrow{\sim} \mathbf{R}\Gamma(G_v, I^n/I^{n+1} \hat{\otimes}_{R}^{\mathbf{L}} T)$$

in $\mathbf{D}(\mathrm{Mod}_{\Lambda})$.

Proof. Let P be a resolution of I^n/I^{n+1} consisting of projective objects in $\mathcal{C}_{\Lambda-\Lambda}$. Then there is a quasi-isomorphism

$$\alpha\colon P \longrightarrow [I^{n+1} \to I^n]$$

of complexes of compact $\Lambda \otimes_R \Lambda^{\circ}$-modules which lifts the identity map on I^n/I^{n+1}. By Lemmas 2.4.3 and 4.4.2(a) the map α induces a quasi-isomorphism

$$P \otimes_{\Lambda} \Lambda_{\chi} \otimes_R T \longrightarrow [I^{n+1} \to I^n] \otimes_{\Lambda} \Lambda_{\chi} \otimes_R T$$

of bounded above complexes in $\mathcal{C}_{R,G_v}$, proving the first isomorphism upon application of Lemma 2.4.6. The second isomorphism is immediate from Lemmas 3.1.1 and 3.1.2. $\square$

Lemma 4.4.6. *For each n, there is a commutative diagram*

$$\begin{tikzcd}
I^n/I^{n+1} \otimes_R^{\mathbf{L}} \mathbf{R}\Gamma(G_v, T) \arrow[r] \arrow[d, "\wr"'] & I^n/I^{n+1} \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R\Big(\mathbf{R}\Gamma(G_v, T^*(1)), \omega_R\Big)[-2] \arrow[d, "\wr"] \\
\mathbf{R}\Gamma(G_v, I^n/I^{n+1} \hat{\otimes}_R^{\mathbf{L}} T) \arrow[r] \arrow[dd, "\wr"'] & \mathbf{R}\mathrm{Hom}_R\Big(\mathbf{R}\Gamma(G_v, T^*(1)), I^n/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R\Big)[-2] \arrow[d, "\wr"] \\
 & \mathbf{R}\mathrm{Hom}_R\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_{\Gamma}(T^*)^{\iota}(1)) \otimes_{\Lambda}^{\mathbf{L}} R, I^n/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R\Big)[-2] \arrow[d, "\wr"] \\
\mathbf{R}\Gamma(G_v, \mathscr{F}_{I^n/I^{n+1}}(T)) \arrow[r] & \mathbf{R}\mathrm{Hom}_{\Lambda^{\circ}}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_{\Gamma}(T^*)^{\iota}(1)), I^n/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R\Big)[-2],
\end{tikzcd}$$

where the vertical morphisms are isomorphisms in $\mathbf{D}(\mathrm{Mod}_R)$.

Proof. By Lemma 4.4.2(b), we may choose a bounded resolution Q of I^n/I^{n+1} by finitely generated projective R-modules. By [Nek, Proposition 3.4.4], we have an isomorphism of complexes

$$\alpha\colon Q \otimes_R C(G_v, T) \longrightarrow C(G_v, Q \otimes_R T)$$

that fits into the commutative diagram

$$\begin{tikzcd}
Q \otimes_R C(G_v, T) \otimes_R C(G_v, T^*(1)) \arrow[r] \arrow[d, "\alpha\otimes\mathrm{id}"'] & Q \otimes_R C(G_v, J_R(1)) \arrow[d, "\alpha'"] \\
C(G_v, Q \otimes_R T) \otimes_R C(G_v, T^*(1)) \arrow[r] & C(G_v, Q \otimes_R J_R(1)),
\end{tikzcd}$$

where α' is defined analogously to α and is also an isomorphism. Since J_R is a bounded complex of R-injectives, we may find homotopy inverses to the quasi-isomorphisms q_R and q_Q of Lemma 4.2.1 that fit in a commutative diagram

$$\begin{tikzcd}
Q \otimes_R C(G_v, J_R(1)) \arrow[r] \arrow[d, "\alpha'"] & Q \otimes_R \tau^{II}_{\geq 2} C(G_v, J_R(1)) \arrow[r] \arrow[d] & Q \otimes_R J_R[-2] \arrow[d, equal] \\
C(G_v, Q \otimes_R J_R(1)) \arrow[r] & \tau^{II}_{\geq 2} C(G_v, Q \otimes_R J_R(1)) \arrow[r] & Q \otimes_R J_R[-2].
\end{tikzcd}$$

We obtain the top commutative square in the lemma and the fact that the vertical morphisms therein are isomorphisms, the one on the right by Lemma 2.1.8 (noting Proposition 4.1.2).

Let P be a resolution of I^n/I^{n+1} consisting of finitely generated projective Λ°-modules. We may view Q as a resolution of Λ°-modules via the augmentation map $\Lambda \to R$. Then, by Proposition 2.4.6, the map

$$P \otimes_\Lambda \mathscr{F}_\Gamma(T) = P \otimes_\Lambda \Lambda_\chi \otimes_R T \longrightarrow Q \otimes_\Lambda \Lambda_\chi \otimes_R T \cong Q \otimes_R T$$

is a quasi-isomorphism of complexes of objects in $\mathcal{C}_{R,G_v}$, and we let f denote its induced map on cochains. Let L be a resolution of R consisting of finitely generated projective Λ-modules. Then, by an opposite version of Theorem 3.1.3, we have a morphism

$$g\colon C(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)) \otimes_\Lambda L \longrightarrow C(G_v, T^*(1))$$

of complexes of R-modules which is a quasi-isomorphism. Noting Lemma 3.3.3, we have a commutative diagram

$$\begin{tikzcd}
C(G_v, P \otimes_\Lambda \mathscr{F}_\Gamma(T)) \otimes_R C(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)) \otimes_\Lambda L \arrow[r] \arrow[d, "f\otimes g"'] & C(G_v, P \otimes_R J_R(1)) \otimes_\Lambda L \arrow[d, "\varepsilon"] \\
C(G_v, Q \otimes_R T) \otimes_R C(G_v, T^*(1)) \arrow[r] & C(G_v, Q \otimes_R J_R(1)),
\end{tikzcd}$$

where ε is induced by the augmentation $L \to R$ and the map $P \to Q$. Taking adjoints and applying the homotopy inverse to q_Q as above, we obtain the commutative diagram

$$\begin{tikzcd}
C(G_v, Q \otimes_R T) \arrow[r] & \mathrm{Hom}_R\Big(C(G_v, T^*(1)), Q \otimes_R J_R\Big)[-2] \arrow[d] \\
C(G_v, P \otimes_\Lambda \mathscr{F}_\Gamma(T)) \arrow[u] \arrow[r] \arrow[d, equal] & \mathrm{Hom}_R\Big(C(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)) \otimes_\Lambda L, Q \otimes_R J_R\Big)[-2] \arrow[d, "\wr"] \\
C(G_v, P \otimes_\Lambda \mathscr{F}_\Gamma(T)) \arrow[r] & \mathrm{Hom}_{\Lambda^\circ}\Big(C(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \mathrm{Hom}_R(L, Q \otimes_R J_R)\Big)[-2],
\end{tikzcd}$$

which yields upon passage to the derived category the lower part of the diagram in the statement of the lemma, the vertical morphisms therein being isomorphisms by Theorem 3.1.3, Lemma 4.4.5 and Lemma 2.1.6. □

Lemma 4.4.7. *The morphisms*

$$\mathbf{R}\Gamma(G_v, \mathscr{F}_{I^n/I^{n+1}}(T)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), I^n/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R\Big)[-2]$$

in $\mathbf{D}(\mathrm{Mod}_\Lambda)$ *are isomorphisms for every* $n \geq 0$.

Proof. By Corollary 2.1.3, it suffices to show that the above morphism is an isomorphism in $\mathbf{D}(\mathrm{Mod}_R)$. The morphism

$$\mathbf{R}\Gamma(G_v, T) \longrightarrow \mathbf{R}\mathrm{Hom}_R\Big(\mathbf{R}\Gamma(G_v, T^*(1)), \omega_R\Big)[-2]$$

in $\mathbf{D}(\mathrm{Mod}_R)$ is an isomorphism by [Nek, Proposition 5.2.4(ii)], and so the top morphism of the diagram in Lemma 4.4.6 is an isomorphism. Since all the vertical morphisms in the diagram are isomorphisms, it follows that the bottom morphism is also an isomorphism, as required. □

Proposition 4.4.8. *The morphisms*

$$\mathbf{R}\Gamma(G_v, \mathscr{F}_{\Lambda/I^n}(T)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda/I^n \otimes_R^{\mathbf{L}} \omega_R\Big)[-2]$$

in $\mathbf{D}(\mathrm{Mod}_\Lambda)$ *are isomorphisms for every* $n \geq 1$.

Proof. As seen in the above discussion, the preceding lemma allows us to perform an inductive argument using the morphism of exact triangles in Proposition 4.4.4 to obtain the required conclusion. □

We finish the proof of Theorem 4.2.2 by passing to the inverse limit.

Proof of Theorem 4.2.2. By Proposition 4.1.2, there exists a quasi-isomorphism

$$W \longrightarrow C(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)).$$

with W a bounded above complex of finitely generated projective Λ°-modules. Since J_R has cohomology groups which are finitely generated over R, Lemma 2.3.1 implies the existence of a subcomplex C of J_R such that C is a complex of finitely generated R-modules and the inclusion $i : C \hookrightarrow J_R$ is a quasi-isomorphism. We fix such a C and write X_n for $[I^n \to \Lambda]$. The complex $\mathrm{Hom}_{\Lambda^\circ}(W, X_n \otimes_R C)$ represents

$$\mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda/I^n \otimes_R^{\mathbf{L}} \omega_R\Big),$$

and $\mathrm{Hom}_{\Lambda^\circ}(W, \Lambda \otimes_R C)$ represents

$$\mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R^{\mathbf{L}} \omega_R\Big),$$

since X_n is a complex of flat R-modules by Lemma 4.4.2(a). Now, for each n, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T)) \arrow[r] \arrow[d] & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R^{\mathbf{L}} \omega_R\Big)[-2] \arrow[d] \\
\mathbf{R}\Gamma(G_v, \mathscr{F}_{\Lambda/I^n}(T)) \arrow[r] & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda/I^n \otimes_R^{\mathbf{L}} \omega_R\Big)[-2]
\end{tikzcd}$$

which induces a commutative diagram

$$\begin{tikzcd}
H^i(G_v, \mathscr{F}_\Gamma(T)) \arrow[r] \arrow[d] & H^{2-i}(\mathrm{Hom}_{\Lambda^\circ}(W, \Lambda \otimes_R C)) \arrow[d] \\
H^i(G_v, \mathscr{F}_{\Lambda/I^n}(T)) \arrow[r] & H^{2-i}(\mathrm{Hom}_{\Lambda^\circ}(W, X_n \otimes_R C))
\end{tikzcd}$$

of cohomology groups. Since this diagram is compatible with n, we obtain the commutative diagram

$$\begin{tikzcd}
H^i(G_v, \mathscr{F}_\Gamma(T)) \arrow[r] \arrow[d] & H^{2-i}(\mathrm{Hom}_{\Lambda^\circ}(W, \Lambda \otimes_R C)) \arrow[d] \\
\varprojlim_n H^i(G_v, \mathscr{F}_{\Lambda/I^n}(T)) \arrow[r] & \varprojlim_n H^{2-i}(\mathrm{Hom}_{\Lambda^\circ}(W, X_n \otimes_R C)).
\end{tikzcd}$$

It remains to show that the upper horizontal map in the latter diagram is an isomorphism. By Proposition 4.4.8, we know that the lower horizonal map is one. Noting that the maps $X_{n+1} \otimes_R C \to X_n \otimes_R C$ are injections of complexes with intersection $\Lambda \otimes_R C$ over all n, we have an isomorphism

$$\varprojlim_n \mathrm{Hom}_{\Lambda^\circ}(W, X_n \otimes_R C) \cong \mathrm{Hom}_{\Lambda^\circ}(W, \Lambda \otimes_R C)$$

of complexes of finitely generated Λ-modules. Since inverse limits are exact for finitely generated Λ-modules (being that they are compact), after taking cohomology groups, we have that the vertical map on the right is an isomorphism. On the other hand, we also have an isomorphism

$$\varprojlim_n (X_n \otimes_\Lambda \mathscr{F}_\Gamma(T)) \cong \mathscr{F}_\Gamma(T)$$

of complexes of objects in $\mathcal{C}^{\Lambda-\mathrm{ft}}_{\Lambda,G_v}$ and hence an isomorphism

$$\varprojlim_n C(G_v, X_n \otimes_\Lambda \mathscr{F}_\Gamma(T)) \cong C(G_v, \mathscr{F}_\Gamma(T)).$$

By [Lim2, Proposition 2.2.10], we then have

$$\varprojlim_n H^j(G_v, X_n \otimes_\Lambda \mathscr{F}_\Gamma(T)) \cong H^j(G_v, \mathscr{F}_\Gamma(T)).$$

Therefore, the vertical map on the left is also an isomorphism. Hence, the top map is an isomorphism, as required. □

4.5 Duality over global fields

We end this paper by describing the global analog of Theorem 4.2.2 that is our main result. Let T be a bounded complex of objects in $\mathcal{C}^{R-\mathrm{ft}}_{R,G_{F,S}}$, and we choose T^* in $\mathcal{C}^{R-\mathrm{ft}}_{R,G_{F,S}}$ as in Section 3.3. As in [Nek, (5.3.3)], we define two morphisms of complexes of $\Lambda \otimes_R \Lambda^\circ$-modules

$$\begin{aligned} {}_c\cup\colon C_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T)) \otimes_R C(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)) &\longrightarrow C_{(c)}(G_{F,S}, \Lambda \otimes_R J_R(1)) \\ \cup_c\colon C(G_{F,S}, \mathscr{F}_\Gamma(T)) \otimes_R C_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)) &\longrightarrow C_{(c)}(G_{F,S}, \Lambda \otimes_R J_R(1)) \end{aligned}$$

which are given by the formulas

$$\begin{aligned} (a, a_S)\, {}_c\cup\, b &= (a \cup b, a_S \cup_S \mathrm{res}_S(b)) \\ a \cup_c (b, b_S) &= (a \cup b, (-1)^{\bar{a}} \mathrm{res}_S(a) \cup_S b_S), \end{aligned}$$

where $\bar{a}$ denotes the degree of a, the direct sum of the restrictions to the primes in S is denoted res_S, the symbol $\cup$ is the total cup product

$$C(G_{F,S}, \mathscr{F}_\Gamma(T)) \otimes_R C(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)) \longrightarrow C(G_{F,S}, \Lambda \otimes_R J_R(1))$$

of Lemma 3.3.3, and $\cup_S$ is the direct sum of the corresponding local cup products. Much as in Lemma 4.2.1 (but now analogously to [Nek, Lemma 5.7.3]), we have a quasi-isomorphism

$$\Lambda \otimes_R J_R \longrightarrow \tau^{II}_{\geqslant 3} C_{(c)}(G_{F,S}, \Lambda \otimes_R J_R(1))$$

of complexes of $\Lambda \otimes_R \Lambda^\circ$-modules, which allows us to use adjoints to define the morphisms in the following theorem.

Theorem 4.5.1. *Let T be a bounded complex of objects in $\mathcal{C}^{R-\mathrm{ft}}_{R,G_{F,S}}$. Then we have an isomorphism*

$$\begin{tikzcd}
\mathbf{R}\Gamma_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T)) \arrow[r, "\sim"] \arrow[d] & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-3] \arrow[d] \\
\mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_\Gamma(T)) \arrow[r, "\sim"] \arrow[d] & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-3] \arrow[d] \\
\bigoplus_{v\in S} \mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T)) \arrow[r, "\sim"] & \bigoplus_{v\in S} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-2]
\end{tikzcd}$$

of exact triangles in $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$.

Proof. The morphism of exact triangles can be constructed as in [Lim2, Theorem 3.2.6]. We remark that it suffices to prove that any two of the three horizontal morphisms is an isomorphism, and we focus on the first two. By an analogous result to Proposition 4.3.2, we may assume that R is a $\mathbb{Z}_p$-flat. That the second of the morphisms is an isomorphism follows as in proof of Theorem 4.2.2. In fact, aside from the obvious changes of notation for cochains and other objects, there are no significant changes to the proof. Therefore, it suffices to prove that the first of the morphisms is also an isomorphism. The only notable differences in this case occur in the proof of Lemma 4.4.6 and in the passage to the limit.

In Lemma 4.4.6, everything goes through as before, except that in order to apply Lemma 2.1.8 to see that the upper right vertical arrow in the diagram of the lemma is a quasi-isomorphism, one must exhibit a quasi-isomorphism $D \to C(G_{F,S}, T^*(1))$

with D a q-projective complex of finitely generated R-modules, which is less obvious as $C(G_{F,S}, T^*(1))$ need not have bounded cohomology. However, as is seen in [Spa, Lemma 3.3] (see also [Kel, Appendix]), such a complex may be constructed as a direct limit $\varinjlim D_n$ using maps $\alpha_n \colon D_n \to D_{n+1}$, that are split injective in each degree, and where $E_n = D_n/\alpha_{n-1}(D_{n-1})$ for $n \geq 1$ is chosen to be quasi-isomorphic to

$$\mathrm{Cone}\Big(\tau^{II}_{\leq n-1} C(G_{F,S}, T^*(1)) \longrightarrow \tau^{II}_{\leq n} C(G_{F,S}, T^*(1))\Big),$$

which has bounded, R-finitely generated cohomology. Since R has finite global dimension, we may choose the E_n as bounded complexes of finitely generated projective R-modules such that, for each k, only finitely many E_n^k are nonzero, which is what was needed.

In passing to the limit, again there exists a q-projective complex $W = \varinjlim W_m$ of Λ°-modules and a quasi-isomorphism $W \to C(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1))$ that is a direct limit of quasi-isomorphisms

$$W_m \longrightarrow \tau^{II}_{\leq m} C(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1))$$

using maps $\iota_m \colon W_m \to W_{m+1}$ for $m \geq 0$ that are split injective in each degree, and the $V_m = W_m/\iota_{m-1}(W_{m-1})$ for $m \geq 1$ and $V_0 = W_0$ are bounded above complexes of finitely generated projective Λ°-modules. The complex $\mathrm{Hom}_{\Lambda^\circ}(W, X_n \otimes_R C)$ represents

$$\mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda/I^n \otimes^{\mathbf{L}}_R \omega_R\Big),$$

by Lemma 2.1.2. By the same reasoning as before, we have

$$\varprojlim_n \mathrm{Hom}_{\Lambda^\circ}(W, X_n \otimes_R C) \cong \mathrm{Hom}_{\Lambda^\circ}(W, \Lambda \otimes_R C).$$

While the terms of W are not necessarily themselves finitely generated over Λ°, we have

$$W^k \cong \bigoplus_{m=0}^{\infty} V_m^k$$

for all k. For every k and l, we then have

$$\mathrm{Hom}_{\Lambda^\circ}(W^k, (X_n \otimes_R C)^l) \cong \prod_{m=0}^{\infty} \mathrm{Hom}_{\Lambda^\circ}(V_m^k, (X_n \otimes_R C)^l),$$

Being products of finitely generated Λ-modules, these terms are compact under the product topology. Moreover, the coboundaries in the complex are continuous with respect to

this topology, as are the maps between terms of the complexes for different n. As inverse limits are exact in the category of compact Λ-modules, we have

$$\varprojlim_n H^{3-i}(\mathrm{Hom}_{\Lambda^\circ}(W, X_n \otimes_R C)) \cong H^{3-i}\Big(\varprojlim_n \mathrm{Hom}_{\Lambda^\circ}(W, X_n \otimes_R C)\Big).$$

The rest of the proof is then as before, with the isomorphism

$$\varprojlim_n H^j_{(c)}(G_{F,S}, X_n \otimes_\Lambda \mathscr{F}_\Gamma(T)) \cong H^j_{(c)}(G_{F,S}, \mathscr{F}_\Gamma(T))$$

now following from the analogous results to it for global and local cohomology, using the definition of compactly supported cohomology as a cone. $\square$

We end with three remarks.

Remark 4.5.2. There is also an analogous diagram for the other adjoint, with morphisms as in the second map in Theorem 4.2.2, and this follows, for instance, by a nearly identical argument.

Remark 4.5.3. For a nonarchimedean prime v in S, the map

$$\tau^{II}_{\geq 2} C(G_v, \Lambda \otimes_R J_R(1)) \longrightarrow \tau^{II}_{\geq 2}\Big(C_{(c)}(G_{F,S}, \Lambda \otimes_R J_R(1))[1]\Big)$$

is a quasi-isomorphism that induces the identity maps on the cohomology groups of each complex. Under this identification, the map on the v-summands in the lower horizontal isomorphism in Theorem 4.5.1 is exactly the first isomorphism of Theorem 4.2.2.

Remark 4.5.4. As a corollary of Theorem 4.5.1, in the case $p = 2$ and $v \in S_\mathbb{R}$, we have that the morphisms

$$\widehat{\mathbf{R}\Gamma}(G_v, \mathscr{F}_\Gamma(T)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\widehat{\mathbf{R}\Gamma}(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-2]$$
$$\widehat{\mathbf{R}\Gamma}(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda}\Big(\widehat{\mathbf{R}\Gamma}(G_v, \mathscr{F}_\Gamma(T)), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-2]$$

in $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_\Lambda)$ and $\mathbf{D}_{\Lambda-\mathrm{ft}}(\mathrm{Mod}_{\Lambda^\circ})$, respectively, are isomorphisms. Identifying $G_\mathbb{R}$ with G_v, these isomorphisms are easily seen to be independent of F and v.

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