INTRODUCTION TO QUANTUM CHAOLOGY

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Abstract. The purpose of this paper is to inform the reader of the topic of quantum chaology and related research areas. Specifically the study of eigenvalues and eigenstates of a quantum Hamiltonian with a classically chaotic counterpart. We say quantum chaology and not quantum chaos since we will be analyzing a chaotic dynamical system in a classical mechanical setting and examining the properties of the analogous quantum system. A dilemma we have is that a chaotic dynamical system is nonlinear while the quantum Hamiltonian of the dynamical system is linear and thus can not be chaotic. Thus the chaos must lie in the properties of wave functions and energy states. The study of energy states in [1] and [2] of classically chaotic systems have led to interesting correlations to eigenvalues of randomly generated symmetric matrices and the Riemann Hypothesis which we introduce later in the paper.

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Contents

1. Introduction

Given a classically chaotic system given by a mapping or differential equation we want to to find its associated quantum system defined by a linear operator which we call the quantum Hamiltonian. We assume that the spectrum of this operator is discrete, and that this quantum Hamiltonian should generate the chaotic classical Hamiltonian in the semi classical limit, that is as Plank's constant $\hbar \to 0$. The chaotic dynamical system is generally nonlinear and the quantum Hamiltonian will be linear and thus can not have chaos. How does the quantum operator display the chaos seen in the classically chaotic system? The chaos must be displayed in the eigenstates. We ask the question, for a specific chaotic system, namely the stadium billiard, how does one go about finding the spectral properties of the associated quantum Hamiltonian? For an integrable system, meaning a system that does not display chaos, the WKB (section (5.1)) expansion leads to approximations for the eigenvalues of the system of any order. For a chaotic system we must use the Gutzwiler trace formula (6.4) to gain information about the eigenvalues. First we give a brief introduction to classical mechanics stressing that a complete knowledge of each subject is not needed to gain a suitable understanding of the task at hand. We introduce the stadium billiard problem and discuss some interesting properties. Then a quick description of the mathematical structure of quantum mechanics is discussed which will lead to analyzing the billiard problem in this setting. Then we give two techniques that are commonly used in the semiclassical domain. Finally we give some results from [3] and [4] on how to compute the eigenvalues of the quantum operator for the stadium billiard. Then we introduce the connection between the eigenvalues of the quantum operator of a classically chaotic system and the eigenvalues of a random matrix. From this we can look at the application of quantum chaology to the Riemann hypothesis. In the end we discuss further work in this area and a response to a question during the presentations on 15, May 2009.

2. Classical Mechanics

2.1. Coordinates.

Throughout the paper we use what is referred to as the canonical coordinates for position and momentum denoted $\mathbf{q}$ and $\mathbf{p}$ respectively. These are vectors describing the position and movement of an object. For example, a particle in three space in Cartesian coordinates would be described by $\mathbf{q} = (x, y, z)^T$, where the ^{T} is the standard transpose. For the sake of generalization throughout this paper we assume that $\mathbf{q} = (q_1, q_2, ..., q_n)^T$ and likewise $\mathbf{p}$. The time derivatives of the canonical coordinates will be denoted $\dot{\mathbf{q}} = \partial \mathbf{q}/\partial t$.

2.2. The Hamiltonian.

Consider a particle of mass m and velocity v in a potential field, call it $V(\mathbf{q})$. Then at any instant, the energy can be written as

$$E = \frac{1}{2}mv^2 + V(\mathbf{q})$$

and the force is given by the negative gradient of the potential, that is

$$F = -\nabla V$$

where $\nabla V = (\partial V/\partial q_1, \partial V/\partial q_2, ..., \partial V/\partial q_n)^T$. The linear momentum $\mathbf{p}$ is given as

$$\mathbf{p} = mv.$$

Now we can rewrite the energy above as

$$H(\mathbf{q},\mathbf{p}) = \frac{1}{2m}\mathbf{p}^2 + V(\mathbf{q}).$$

This is an example of a Hamiltonian function which is a way to write the energy of a system as a function of position and momentum. The Hamiltonian equations of motion are given by

$$\begin{aligned} \frac{\partial q_i}{\partial t} &= \frac{\partial H}{\partial p_i} \\ \frac{\partial p_i}{\partial t} &= -\frac{\partial H}{\partial q_i}. \end{aligned} \tag{2.1}$$

For all $i = 1, 2, ..., n$.

2.3. **Lagrangian Mechanics.**

Definition 2.2. Let C be a curve in the plane with $\mathbf{q}(t)$, $t_0 \leq t \leq t_1$, and $L = L(\dot{\mathbf{q}}, \mathbf{q}, t)$ be a differentiable function. Then the *action integral* is defined as

$$\Phi(C) = \int_{t_0}^{t_1} L(\dot{\mathbf{q}}, \mathbf{q}, t)\, dt.$$

There are a number of theorems that are important in classical mechanics dealing with the action integral. Some of them will be stated and used without proof.

Theorem 2.3. *The curve γ is an extremal of the function $\Phi(\gamma)$ on the space of curves passing through the two points $\mathbf{q}(t_0)$ and $\mathbf{q}(t_1)$, if and only if the following equation is satisfied*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0.$$

These equations are called the Euler-Lagrange equations for the functional Φ. γ is called a motion in the Lagrangian system with the Lagrangian function L.

2.3.1. *Hamilton's principle of least action.*

Theorem 2.4. *Given a Hamiltonian function H and let γ be the motion of the system. Then the extremal of the action integral is satisfied with*

$$\Phi(C) = \int_{t_0}^{t_1} L\, dt, \quad \textit{where } L = T - U.$$

T and U are the kinetic and potential energies respectively.

The action integral Φ depends on time t. If we wish to replace t with energy E we define a new kind of action integral and define it as

$$S(\mathbf{q}(t_0), \mathbf{q}(t_1), E) = \Phi(C) + Et. \tag{2.5}$$

From this definition, the relationships between S and the momentum at $\mathbf{p}(t_0)$ and $\mathbf{p}(t_1)$ are

$$p_i(t_1) = \frac{\partial S}{\partial q_i(t_1)}, \quad p_i(t_0) = -\frac{\partial S}{\partial q_i(t_0)}, \quad t = \frac{\partial S}{\partial E}.$$

The action integral S is later used in section (6.0.2).

The subject of classical mechanics is a well studied one and we have just introduced basic notions to help in our search for eigenvalues and eigenstates of just one system. If the reader would like to delve into classical mechanics deeper than this paper goes, please see [5], [6], as just a few of the many references on classical mechanics.

3. Billiard Problem

Now we consider the two dimensional billiard problem. As described in [4], a billiard is a two-dimensional planar domain, Ω, where point particles move with constant velocity, in straight line orbits between the boundary $\partial\Omega$, and the domain. The boundary preserves specular reflection, angle of incidence equals angle of reflection, and we assume conservation of energy.

3.1. **Stadium Billiard.**

The billiard problem of interest in this paper is the stadium billiard. The boundary consists of two parallel line segments of length d connecting two semicircles of radius r. Consider a point particle with initial condition $\mathbf{q}_0 = (x_0, y_0) \in \text{int}(\Omega)$ and initial velocity $\mathbf{v}_0$. The potential $V(\mathbf{q}) = 0$ in the domain Ω and the potential $V(\mathbf{q}) = \infty$ outside the domain. Thus specular reflection is obtained at the boundary. This situation is pictorially described in figure (1).

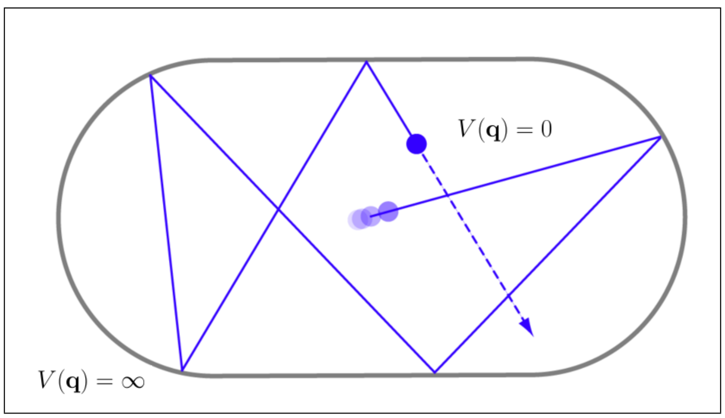

Figure 1. The set up to the stadium billiard problem, given a point particle with initial velocity $\mathbf{v}$, and initial position $\mathbf{q}_0 = (x_0, y_0)$.

We wish to analyze the orbit of this particle. To do so we must have an understanding of what happens to the particle on the boundary. While the particle is in the interior of Ω, before it has reached a boundary point, we may describe the motion as straight lines, or

$$\mathbf{q}(t) = \mathbf{q}_0 + t\mathbf{v}_0.$$

We now consider what happens when the particle meets a boundary point $\mathbf{q}_1 \in \partial\Omega$ at some time $t_1 > 0$.

3.1.1. *Line Segment.*

When $\mathbf{q}_1 \in \partial\Omega$ lies on one of the parallel line segments, specular reflection gives a straight forward calculation. Qualitatively speaking, the speed of the particle remains unchanged while the direction is changed in one component. A new velocity $\mathbf{v}_1$ is given by a linear transformation

$$\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{v}_0.$$

Thus we give the position of the particle after time t_1 traveling in the interior of Ω as

$$\mathbf{q}(t) = \mathbf{q}_1 + \mathbf{v}_1 t, \qquad \text{for all } t \geq t_1 > 0.$$

3.1.2. *Arc Sector.* When $\mathbf{q}_1 \in \partial\Omega$ lies on one of the semicircles, one has a more complicated calculation than the one above. To find the velocity vector after impact, in essence, we must take the initial velocity vector and reflect it over the line that connects the center of the circle, which we will call $\mathbf{r}_1$, with the point of impact $\mathbf{q}_1$. To do so we take advantage of the Househoulder reflection. The reflection matrix is given by

$$\left(I - 2\frac{(\mathbf{r}_1 - \mathbf{q}_1)(\mathbf{r}_1 - \mathbf{q}_1)^*}{r^2} \right),$$

Where $(\mathbf{r}_1 - \mathbf{q}_1)/r$ is the unit vector pointing from the center of the semicircle to the point of impact $\mathbf{q}_1$, and $(\cdot)^*$ is the Hermition transpose, which in this case is just the transpose. Next we must reflect $\mathbf{v}_0$ which gives the new velocity vector of

$$\mathbf{v}_1 = \left(I - 2\frac{(\mathbf{r}_1 - \mathbf{q}_1)(\mathbf{r}_1 - \mathbf{q}_1)^*}{r^2} \right) \mathbf{v}_0.$$

Now we give the position of the particle after time t_1 traveling in the interior of Ω as

$$\mathbf{q}(t) = \mathbf{q}_1 + \mathbf{v}_1 t, \qquad \text{for all } t \geq t_1 > 0.$$

With this information, we now can iteratively give the position and velocity of a point particle as time t progresses. From here on we will give the coordinates as $\mathbf{x} = (\mathbf{q}, \mathbf{v})^T$.

3.2. **Dynamical Systems and Chaos.** To go into any further analysis, we must first discuss some elementary ideas in the fields of dynamical systems and chaos.

Definition 3.1. A *dynamical system* is a vector field in phase space by using the arrows of the vector field to create continuous curves called trajectories. A dynamical system can also be described by a discrete-time mapping.

An example of a dynamical system is a Hamiltonian system $H(\mathbf{p}, \mathbf{q})$, where the vector field is given by the equations of motion (2.1). We also give the stadium billiard as a discrete-time mapping between impacts on the boundary in section (3.3).

Definition 3.2. Given an initial condition $\mathbf{x}_0$, a *periodic orbit* is a trajectory which returns to $\mathbf{x}_0$ at some later time. A *primitive periodic orbit* is the shortest such periodic orbit.

Example 3.3. For the stadium billiard, given the initial condition $\mathbf{x}_0 = (\mathbf{q}_0, \mathbf{v}_0)^T = (0, 0, 2/\sqrt{5}, 1/\sqrt{5})^T$, at time $t = 8r$ the particle returns to $\mathbf{x}_0$. This orbit is not primitive since the particle was at $\mathbf{x}_0$ at times $t = 2r$, $t = 4r$, and $t = 6r$ as well.

Definition 3.4. A *chaotic orbit*, on the other hand, is one which never returns to its initial condition.

Now we give a rough description of two types of dynamical systems.

Definition 3.5. A dynamical system is *strongly chaotic* if almost every initial condition yields a chaotic orbit which comes arbitrarily close to any given point in phase space. That is, the space of initial conditions is dense in the domain Ω.

Definition 3.6. A dynamical system is *completely integrable* if it does not have any chaotic orbits.

3.3. **A Mapping of Coordinates.** The formulas found in section (3.1) give a step by step method of finding the position of the point particle given the initial position and velocity. The particle travels in straight lines between the boundary so it is important to think of the movement of the particle in a nontrivial manner. That is the movement of the particle in terms of the boundary. We can do so by introducing a mapping such as in [7].

Definition 3.7. A *Poincaré map M* is a dynamical system written as

$$\mathbf{x}_{n+1} = M(\mathbf{x}_n).$$

Given a particle on the boundary we can uniquely determine its position and velocity (by assuming unit velocity) by two distinct coordinates. Now we aim at finding a linear mapping,

$$\Psi : [0, 2\pi) \times [0, 2\pi) \to [0, 2\pi) \times [0, 2\pi),$$

Such that given a particle that bounces n times off the boundary we can find the $n+1$ point of impact with the boundary, denoted $\mathbf{x}_{n+1}$, by taking the initial condition $\mathbf{x}_0$ and applying the mapping

$$\Psi_n(\Psi_{n-1}(\Psi_{n-1}(...(\Psi_1 \mathbf{q}_0)))) = \mathbf{q}_{n+1},$$

where Ψ_i is the map for the i^{th} bounce.

3.3.1. *Line Segment.* Given a set of coordinates, $\mathbf{x} = (b, \theta)^T$ where $b \in [0, 2\pi]$ tells us where on the boundary the particle is, by using an angle measurement between the x-axis and a line through the center of the stadium and the point on the boundary, and $\theta \in [0, 2\pi]$, is the angle of velocity with respect to the $x-$axis, we wish to find a function $\Psi(b_n, \theta_n)^T = (b_{n+1}, \theta_{n+1})^T$ where $(b_{n+1}, \theta_{n+1})^T$ is the next instance when the particle reaches the boundary. This coordinate system is depicted in figure (2). We have two cases for Ψ, call them Ψ_l and Ψ_a for when the particle impacts the line and arc respectively. First we calculate Ψ_l. By using planar geometry, and given the coordinates (b_n, θ_n) we can find b_{n+1} and θ_{n+1},

$$b_{n+1} = \arcsin\left(\frac{\sin(\pi - b_n - \theta_n)}{s_3}\right) s_2$$

where

$$s_1 = r/\sin(b_n), \quad s_2 = \sqrt{d^2(\cos^2(\theta_n) + 1)}, \quad s_3 = s_1^2 + s_2^2 - 2 s_1 s_2 \cos(\pi - b_n - \theta).$$

the complexity of b_{n+1} is reduced in the θ argument to,

$$\theta_{n+1} = \theta_n + \pi/2.$$

Thus

$$\mathbf{x}_{n+1} = \Psi_l \begin{pmatrix} b_n \\ \theta_n \end{pmatrix} = \begin{pmatrix} \arcsin\left(\frac{\sin(\pi - b_n - \theta)}{s_3}\right) s_2 \\ \theta_n + \pi/2 \end{pmatrix}. \tag{3.8}$$

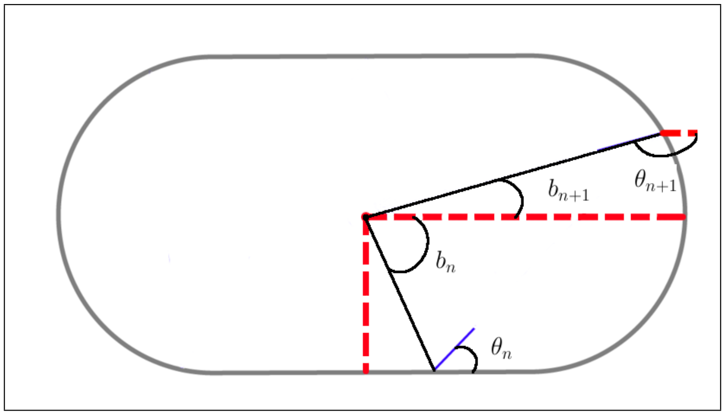

FIGURE 2. A pictorial description of the mapping of the particle from the initial point $x_n = (b_n, \theta_n)^T$ to the next point of impact on the boundary x_{n+1}.

3.3.2. *Arc Segment.* Again for the position of a particle on the boundary and the velocity denoted $(b_n, \theta_n)^T$, we want to find $(b_{n+1}, \theta_{n+1})^T$ for a particle reflecting off one of the semicircles of the stadium billiard. Again this reduces to a geometry problem with solving triangles.

$$b_{n+1} = \arccos\left(\frac{s_1^2 + s_3^2 - s_2^2}{2s_1 s_2}\right),$$

where

$$s_1 = r/\sin(b_n), \quad s_2 = s_1\frac{\sin(b_n)}{\sin(\theta_n)} - r\frac{\sin(\arcsin((d-s_3)\sin(\pi-\theta_n)/r) + \theta_n)}{\sin(\theta_n)},$$

$$s_3 = \sqrt{s_1^2 + s_2^2 - 2s_1 s_2 \cos(\pi - b_n - \theta_n)}.$$

For the new velocity we again get a simpler expression,

$$\theta_{n+1} = 2r\frac{\sin(\arcsin((d-s_3)\sin(\pi-\theta_n)/r)}{\sin(\theta_n)} + (\pi - \theta_n).$$

Which gives the mapping

$$\mathbf{x}_{n+1} = \Psi_a\begin{pmatrix} b_n \\ \theta_n \end{pmatrix} = \begin{pmatrix} \arccos\left(\frac{s_1^2+s_3^2-s_2^2}{2s_1s_2}\right) \\ 2r\frac{\sin(\arcsin((d-s_3)\sin(\pi-\theta_n)/r)}{\sin(\theta_n)} + (\pi-\theta_n) \end{pmatrix}. \tag{3.9}$$

Definition 3.10. The *linear stability matrix* of the Poincaré map, M, is denoted DM and is the Jacobian of the Poincaré map.

Example 3.11. Consider the stadium billiard with line segment of length $d = 2$ and radius $r = 1$. Then it can be shown that given the initial condition $\mathbf{x}_0 = (0, 0, 2/\sqrt{5}, 1/\sqrt{5})^T$, this results in a periodic orbit. That is

$$\Psi_4\mathbf{x}_0 = \mathbf{x}_0.$$

Using the notation of the Poincaré mappings found in equations (3.8) and (3.9) we have

$$\Psi_4\mathbf{x}_0 = \Psi_l(\Psi_a(\Psi_l(\Psi_a(\mathbf{x}_0)))) = \mathbf{x}_0.$$

It will be convenient to think of this primitive periodic orbit as an element of the set {p.o.} which is the set of all periodic orbits (primitive and non) for the stadium billiard. Thus we shall call this orbit $\ell \in \{\text{p.o.}\}$. Now if we want to find the linear stability matrix of this periodic orbit we would have to find the Jacobian for Ψ_a and Ψ_l, that is

$$J(\Psi) = \begin{pmatrix} \frac{\partial b_{n+1}}{\partial b_n} & \frac{\partial b_{n+1}}{\partial \theta_n} \\ \frac{\partial \theta_{n+1}}{\partial b_n} & \frac{\partial \theta_{n+1}}{\partial \theta_n} \end{pmatrix}$$

We now can find the Jacobian for Ψ_4 as

$$J(\Psi_4) = J(\Psi_l)J(\Psi_a)J(\Psi_l)J(\Psi_a),$$

and evaluate this end result at $\mathbf{x}_0$. So we label this as

$$D\Psi^{(\ell)} = J(\Psi_4)|_{\mathbf{x}_0},$$

and call it the linear stability matrix for the periodic orbit $\ell \in \{\text{p.o.}\}$.

Now we analyze this system by using quantum mechanics. First we give an introduction to quantum mechanics.

4. Quantum Mechanics

While observing classical mechanical systems, all the observations can be made with virtually no disturbance to the system. The same can not be said with small particles such as atoms. When one wishes to make a measurement in a quantum system, such as position or momentum, the system must be altered to make the measurement. Thus the measurement is only valid for the state of the system at that moment. This leads to the *uncertainty principle* in quantum mechanics. In short, the uncertainty principle says that to measure an atomic particle's position and momentum within a uncertainty of Δq and Δp respectively, we have the following inequality,

$$\Delta q \Delta p \geq \hbar.$$

$\hbar$ is Plank's constant which can be found experimentally. In the context of quantum chaology, we first analyze the dynamical system in classical mechanics, then look at the dynamics in quantum mechanics. We expect to recover the classical dynamics in the semi classical limit, that is, as $\hbar \to 0$, from the quantum mechanical system. We will call this domain the *semi classical domain*.

4.1. The Postulates of Quantum Mechanics.

The postulates of quantum mechanics were derived by trial and error. They are stated below, in no particular order, to give the reader a working knowledge of quantum mechanics.

Definition 4.1. An *observable* is a property of a system which can be measured by certain physical operations. The observable we work with in this paper is the momentum $\mathbf{p}$.

postulate 4.2. *The first postulate states that an observable A can be assigned an operator, $\hat{A}$, such that when the observable is measured with value a the corresponding equation is satisfied,*

$$\hat{A}\phi = a\phi.$$

ϕ is refered to as the eigenfunction of operator $\hat{A}$ with eigenvalue a.

Example 4.3. In Quantum mechanics the operator that is associated with momentum is given by

$$\hat{\mathbf{p}} = -i\hbar\nabla. \tag{4.4}$$

To find eigenvalues and eigenfunctions, we want to find functions ϕ, such that

$$\hat{\mathbf{p}}\phi = -i\hbar\nabla\phi = p\phi$$

where p are the possible values that measurement of p will yield. Note that p is a parameter not a variable. If there are no boundary conditions on ϕ, then we call the particle a free particle and have the following eigenfunctions

$$\phi(\mathbf{q}) = B\exp\left(\frac{ip\mathbf{q}}{\hbar}\right),$$

where B is some constant. This free particle example will be important in later sections (see section 5.1). Often times, the wave number k, is defined as $k = p/\hbar$ and the above eigenfunction is written as

$$\phi_k = Be^{ik\mathbf{q}}.$$

postulate 4.5. *The measurement a of the observable A leaves the system in the state ϕ_a, where ϕ_a is the eigenvector or often called an eigenstate of $\hat{A}$ with corresponding eigenvalue a.*

postulate 4.6. *The state of a systems can be represented by a continuous and differentiable function called the wave function and denoted as ψ. The properties of ψ is that for any observable A, relevant to the system in the state $\psi(\mathbf{q}_0, t)$ at time t, has the average or expected value*

$$\langle A\rangle = \int \psi^* \hat{A}\psi d\mathbf{q} = \langle\psi|\hat{A}|\psi\rangle.$$

postulate 4.7. *The state of the system ψ develops in time by the time-dependent Schrödinger equation*

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{q}, t) = \hat{H}\psi(\mathbf{q}, t).$$

$\hat{H}$ is the quantum Hamiltonian operator.

4.2. **Billiard Problem in Quantum Mechanics.** We now want to look at the stadium billiard in the quantum setting. By using the notion of the momentum of the particle as being an observable and using equation (4.4), we write the classical Hamiltonian inside the domain Ω (that is where potential is zero),

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2m}\mathbf{p}^2,$$

as the quantum Hamiltonian

$$\hat{H}(\hat{\mathbf{p}}) = \frac{\hbar^2}{2m}\nabla^2.$$

Then by postulate III of quantum mechanics there is a wave function $\psi(\mathbf{q}, t)$ that is continuous and differentiable. Furthermore by postulate IV, $\psi(\mathbf{q}, t)$ develops in time by

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{q}, t) = \frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{q}, t).$$

The potential of the system does not vary with time, thus we can write the wave function as $\psi(\mathbf{q},t) = \phi(\mathbf{q})e^{i\omega t}$, and it can be shown that the following stationary Schrödinger equation is satisfied

$$-\frac{\hbar^2}{2m}\nabla^2\phi(\mathbf{q}) = E\phi(\mathbf{q}). \tag{4.8}$$

This is the eigenvalue problem, where E are the different energies and $\phi(\mathbf{q})$ are the eigenstates for energy E. Now we work to find these energies.

This serves a brief introduction to quantum mechanics that we will need in this paper. We have just scratched the surface of this subject and if the reader wishes to study more refer to [8], [9], [10], [11].

5. Approximations in the Semi Classical Domain

In this section we wish to derive the mathematical foundation of the analogy between the classical and quantum equations of motion. We work in the near classical domain so both quantum and classical mechanics are important. Two approximations are outlined in this section.

5.1. **WKB expansion.** If a completely integrable system was behind the eigenvalue problem of equation (4.8), then all orders of energies could be approximated by the WKB method. The exact calculation of these energies is not given, but a general outline of the WKB method is given for the reader. The WKB expansion, which is named after *Wentzel-Kramers-Brillouin*, is for solutions to Schrödinger's equation which are valid in the semi classical domain. This method can only be used for one-dimensional problems, but can be generalized for higher dimensional systems. While deriving its results we consider the one dimensional variable x. For a potential V which is constant, the wavefunction closely approximates the free particle state as mentioned in section (4.1) and can be written as

$$\phi(x) = Ae^{ikx} = Ae^{ipx/\hbar}.$$

So we expect solutions of the form

$$\phi(x) = Ae^{iS(x)/\hbar}$$

where $S(x)$ is a real-valued function. Plugging this function into the stationary Schrödinger's equation (4.7) with potential $V \neq 0$ yields

$$-i\hbar\frac{\partial^2 S}{\partial x^2} + \left(\frac{\partial S}{\partial x}\right)^2 = 2mE. \tag{5.1}$$

Where E is the energy of the system. We now want to examine what happends when $\hbar \to 0$. We expect that the wave packet ϕ will become the classical particle. Now we expand $S(x)$ with a Taylor Series around $\hbar$ to write,

$$S(x) = S_0(x) + \hbar S_1(x) + \frac{\hbar^2}{2}S_2(x) + ...$$

Now we use this expansion in equation (5.1) and truncate to get

$$0 = -2mE - i\hbar\left(\frac{\partial^2 S_0(x)}{\partial x^2} + \hbar\frac{\partial^2 S_1(x)}{\partial x^2} + O(\hbar^2)\right) + \left(\frac{\partial S_0(x)}{\partial x} + \hbar\frac{\partial S_1(x)}{\partial x} + \frac{\hbar}{2}\frac{\partial S_2(x)}{\partial x} + O(\hbar^3)\right)^2,$$

and

$$0 = \left[\left(\frac{\partial S_0(x)}{\partial x}\right)^2 - 2mE\right] + \hbar\left[2\frac{\partial S_0(x)}{\partial x}\frac{\partial S_1(x)}{\partial x} - i\frac{\partial^2 S_0(x)}{\partial x^2}\right]$$
$$+ \hbar^2\left[\left(\frac{\partial S_1(x)}{\partial x}\right)^2 + \frac{\partial S_0(x)}{\partial x}\frac{\partial S_2(x)}{\partial x} - i\frac{\partial^2 S_1(x)}{\partial x^2}\right] + O(\hbar^3).$$

This equation must be satisfied for all $\hbar$ small. Thus each coefficient of a power of $\hbar$ can be set to zero. This leads to a series of coupled equation which we will list the first two:

$$\left(\frac{\partial S_0}{\partial x}\right)^2 = p^2$$
$$\frac{\partial S_0}{\partial x}\frac{\partial S_1}{\partial x} = \frac{i}{2}\frac{\partial^2 S_0}{\partial x^2}.$$

By integrating the first equation we get

$$\frac{S_0}{\hbar} = \pm\int_{x_0}^{x} k(x)\,dx \qquad \text{where } k = p/\hbar,$$

and using this to integrate the second equation yields

$$S_1 = \frac{i}{2}\ln \hbar k(x).$$

Thus in the semi classical domain, we can write ϕ as

$$\phi(x) = Ak^{-1/2}\exp\left(i\int k\,dx\right) + Bk^{-1/2}\exp\left(-i\int k\,dx\right). \tag{5.2}$$

5.2. **The Stationary Phase Method.** To find the energies for the quantum Hamiltonian of a chaotic dynamical system like the stadium billiard, more work is needed. One important tool in quantum chaology is the method of stationary phase. It is a way of simplifying an integral with a rapidly oscillating integrand. To this end consider the integral

$$\int f(\mathbf{q})\exp(iF(\mathbf{q})/\hbar)\,d\mathbf{q}$$

for some sufficiently smooth function f, and F can be thought of as the action integral (as is done in most applications in quantum chaology). As $\hbar \to 0$, F must remain continuous and further the integrand is a rapidly oscillating function. The integral of a rapidly oscillating function should be small, except for the contribution arising from large contributions in $\mathbf{q}$ producing small variations in F. In other words, F has a first derivative of zero. Consider F in this small region, let's say about $\mathbf{q} = \mathbf{q}_0$, where its first derivative is zero. Consider the expansion about this point,

$$F(\mathbf{q}) = F(\mathbf{q}_0) + F'(\mathbf{q}_0)(\mathbf{q}-\mathbf{q}_0) + \frac{F''(\mathbf{q}_0)}{2}(\mathbf{q}-\mathbf{q}_0)^2 + O((\mathbf{q}-\mathbf{q}_0)^3)$$
$$= F(\mathbf{q}_0) + \frac{F''(\mathbf{q}_0)}{2}(\mathbf{q}-\mathbf{q}_0)^2 + O((\mathbf{q}-\mathbf{q}_0)^2)$$

since $F'(\mathbf{q}_0) = 0$. Now we look at what happens to the integral as $\hbar \to 0$. First we consider the integral

$$\int e^{iF/\hbar}\, d\mathbf{q} \approx \int_{\mathbf{q}_0-\epsilon}^{\mathbf{q}_0+\epsilon} \exp\left(i(F(\mathbf{q}_0) + F''(\mathbf{q}_0)/2(\mathbf{q}-\mathbf{q}_0)^2 + O((\mathbf{q}-\mathbf{q}_0)^3))/\hbar\right)\, d\mathbf{q}.$$

Now we separate the terms and drop the $O((\mathbf{q}-\mathbf{q}_0)^3)$ to approximate the integral and get

$$\exp(iF(\mathbf{q}_0)/\hbar) \int_{\mathbf{q}_0-\epsilon}^{\mathbf{q}_0+\epsilon} f(\mathbf{q}) \exp\left(\frac{iF''(\mathbf{q}_0)(\mathbf{q}-\mathbf{q}_0)^2}{2\hbar}\right)\, d\mathbf{q}.$$

This has the familiar form of the Gaussian Integral, further more, the gaussian is a approximation of the Dirac delta function, and is equal to it with a normalization term in the limit as $\hbar \to 0$. The only part missing is the normalization factor for the integral. We can compute

$$\int \exp\left(\frac{iF''(\mathbf{q}_0)(\mathbf{q}-\mathbf{q}_0)^2}{2\hbar}\right)\, dq$$

by a contour integral in the complex plane, which gives the normalization term. Thus we get that the integral is

$$\int \exp\left(\frac{iF''(\mathbf{q}_0)(\mathbf{q}-\mathbf{q}_0)^2}{2\hbar}\right)\, dq = \sqrt{\left(\frac{2\pi\hbar}{|F''(\mathbf{q}_0)|}\right)} \exp(\pm i\pi/4),$$

where $\exp(\pm i\pi/4)$ is positive for $F''(\mathbf{q}_0) > 0$, and negative for $F''(\mathbf{q}_0) < 0$. Thus we are left with the inequality

$$\left|\exp(iF(\mathbf{q}_0)/\hbar) \int_{\mathbf{q}_0-\epsilon}^{\mathbf{q}_0+\epsilon} f(\mathbf{q}) \exp\left(\frac{iF''(\mathbf{q}_0)(\mathbf{q}-\mathbf{q}_0)^2}{2\hbar}\right)\, d\mathbf{q}\right| \leq \int_{-\infty}^{\infty} f(\mathbf{q}) \exp\left(\frac{-i|F''(\mathbf{q}_0)|(\mathbf{q}-\mathbf{q}_0)^2}{2\hbar}\right)\, d\mathbf{q}$$
$$= \exp(iF(\mathbf{q}_0)/\hbar)\sqrt{\left(\frac{2\pi\hbar}{|F''(\mathbf{q}_0)|}\right)}\, |f(\mathbf{q}_0)\exp(\pm i\pi/2)|\,,$$

in the limit as $\hbar \to 0$. Therefore

$$\int f(\mathbf{q}) e^{iF/\hbar}\, d\mathbf{q} = O(\hbar^{1/2}).$$

Note that we claimed that the integral was made largely of contributions where $F(\mathbf{q})$ had derivative zero. Suppose that we expand $F(\mathbf{q})$ around a point that was not stationary, that is a non-zero derivative. Then dropping all the order two terms we have

$$\int f(\mathbf{q}) \exp(iF(\mathbf{q})/\hbar)\, d\mathbf{q} \approx \int f(\mathbf{q}) \exp(iF(\mathbf{q}_0)/\hbar) \exp(iF(\mathbf{q}_0)(\mathbf{q}-\mathbf{q}_0)/\hbar)\, d\mathbf{q}.$$

Then by the Riemman-Lebesgue lemma, this integral is $O(\hbar)$ and thus goes to zero as $\hbar \to 0$ faster than the prior integral where $\mathbf{q}_0$ was a stationary point. Thus the integral is made up of contributions where $F(\mathbf{q})$ has zero derivative.

6. Eigenstates of the Billiard problem

Now we search for the eigenvalues and eigenstates of the quantum analogue of the classical billiards problem.

6.0.1. *Classical to Quantum Mechanics.* We give a brief description on how classical systems are analyzed in a quantum setting.

Definition 6.1. The *propagator* $K_c(\mathbf{q}_1, \mathbf{q}_0, t)$, is defined as the value of the quantum wave function ψ at $\mathbf{q}_1 = \mathbf{q}(t_1)$ at time $t_1 > 0$, if ψ was concentrated at in $\mathbf{q}_0 = \mathbf{q}(t_0)$ at time $t = t_0$.

The propagator has a explicit formula given by Van Vleck and can be found in ([3], pg 184). We do not write down the explicit formula since we do not use it and it will not give any useful insight to the task at hand.

6.0.2. *Green's function.*

Definition 6.2. The *Green's function* is the propagator in energy space and can be defined as

$$G(\mathbf{q}_1, \mathbf{q}_0, E) = \frac{1}{i\hbar} \int_0^\infty K(\mathbf{q}_1, \mathbf{q}_0, t) \exp(iEt/\hbar)\, dt. \tag{6.3}$$

The Green's function satisfies the equation

$$\left(E + \frac{\hbar^2}{2m}\nabla^2\right) G(\mathbf{q}_1, \mathbf{q}_0, E) = \delta(\mathbf{q}_1 - \mathbf{q}_0).$$

We want a simpler equation for the Green's function in equation (6.3). We can plug Van Vleck's equation for K and get an integral of the form

$$G(\mathbf{q}_1, \mathbf{q}_0, E) = \frac{1}{i\hbar} \int_0^\infty C(\mathbf{q}_1, \mathbf{q}_0, t) \exp(i(\Phi(C) + Et)/\hbar)\, dt,$$

where C are all the curves passing between $\mathbf{q}_1$ and $\mathbf{q}_0$. We can use the method of stationary phase in section (5.2) to simplify this integral. Thus we look for curves where the derivative of $\Phi(C) + Et$ are zero. By Hamilton's Principle, this occurs when C is the actual motion of the classical system. Thus $\Phi(C) + Et$ is the action integral and we can write the Green's function as

$$G_c(\mathbf{q}_1, \mathbf{q}_0, E) = \frac{2\pi}{(2\pi i\hbar)^{(n+1)/2}} \sum_{a\in\sigma} \sqrt{(-1)^{n+1} D} \exp(iS(\mathbf{q}_1, \mathbf{q}_0, E)/\hbar - i\mu\pi/2).$$

Where σ are all straight line orbits in the stadium billiard from $\mathbf{q}_0$ to $\mathbf{q}_1$. The D is given by

$$D(\mathbf{q}_1, \mathbf{q}_0, E) = \frac{1}{|\mathbf{p}_0||\mathbf{p}_1|} \left| \frac{-\partial^2 S}{\partial q_i(t_0) \partial q_j(t_1)} \right|,$$

and the $\mu\pi/2$ can be thought of as different phases of the wave function ψ.

This expression of the Green's function is an approximation (sometimes exact) to the correct quantum mechanical Green's function for a classical system (e.g. free particle system).

6.1. **The Trace Formula.** To try to construct a wave function out of this Green's function is analytically difficult and thus we look at taking the trace to find eigenvalues and eigenfunctions analytically. Now define the integral

$$g_c(E) = \int G_c(\mathbf{q}_0, \mathbf{q}_1, E) dq.$$

Now set $\mathbf{q}_0 = \mathbf{q}_1 = \mathbf{q}$ and all the trajectories in G_c must close. Next, by using the stationary-phase method it can be shown that the calculation of $g_c(E)$ is reduced to the sum over all periodic orbits. Then by using the Gutzwiller approximation to the trace formula [3] we write

$$g_c(E) = \hbar^{-1} \sum_{k\in\{\text{p.o.}\}} \frac{T_k}{(\det|D\Psi^k - I|)^{1/2}} \exp(iS/\hbar - \ell i\pi/2), \tag{6.4}$$

where T_k is the length of the primitive orbit, $D\Psi^k$ is the linear stability matrix for the Poincaré map mentioned in section (3.3), and the action integral is taken over the period orbit k. From this formula, one can extract the eigenvalues and eigenstates for the quantum Hamiltonian to the stadium billiard problem. Notice how the left hand side of equation (6.4) is a purely quantum expression, that is the trace of the Green's function which satisfies the stationary Schrödinger equation. The right hand side of equation (6.4) is calculated by using classical mechanics involving periodic orbits of the stadium billiard, the action integral S, and the linear stability matrix.

7. Applications to Open Problems

We now look at the spectrum $\{E_k\}$ of the quantum Hamiltonian operator $\hat{H}$. For large energy levels E_k we expect the classical dynamics of our system to be important. Thus we look in the semiclassical domain. Namely, the trace of the classical Green's function in equation (6.4).

7.1. **BGS Conjecture.** For a large energy levels say, $E_\lambda \gg 1$, what is the distribution of energies in a window around E_λ at the microscopic scale? The microscopic scale is referred to as the average level spacing between energies is $\delta E_\lambda \sim 2\pi/E_\lambda$. It has been shown that the energies at this microscopic scale are an infinite sequence of 'pseudorandom points'. Given a classically chaotic dynamical system (such as the stadium billiard), the Bohigas-Giannoni-Schmit conjecture claims that this sequence of energies is statistically equivalent to the eigenvalues of a random matrix belonging to one of the three standard Gaussian ensembles. The Essen group (by combinatorics) reproduced the Random Matrix Theory form factor from the Gutzwiller's trace formula (6.4). There are problems that need to be resolved to come to a rigorous proof of the BGS. One is the question of the validity of (6.4) for small energies to which this semiclassical approximation is expected to break down. One insight to take away from the BGS conjecture is that the eigenvalues of a quantum Hamiltonian of a classically chaotic dynamical system mimic the eigenvalues of a self-adjoint random matrix.

7.2. **Riemann Zeta Hypothesis.** Consider the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{7.1}$$

This is the Riemann zeta function, and the Riemann hypothesis states that all non-trivial complex zeros of $\zeta(s)$ have $\text{Re}(s) = 1/2$ and are written as $s_n = 1/2 + it_n$. It is speculated that the t_n appeared to be distributed according to Random Matrix Theory. Instead of working with random matrices, the author of [1] thinks that the t_n could be spectra of a deterministic quantum system with a chaotic classical counterpart, such as the stadium billiard. This seems more intuitive than using Random Matrix Theory since the non-trivial zeros of the Riemann zeta function are not random. Thus the Riemann hypothesis could be proved if the t_n are shown to be eigenvalues of some self-adjoint operator. That is, if a chaotic dynamical system can be found that produces a quantum operator with eigenvalues t_n. So what is the system responsible for the t_n? If one is found then a proof of the Riemann hypothesis would follow. Another connection to the Riemann hypothesis, posed by Connes, is that the Riemann zeros are not spectrum of an operator, rather, gaps in the spectrum.

8. Future Work

This is just a taste of quantum chaology. There is still much work to be done in this field including a million dollar problem in the Riemann hypothesis. The next step in accordance with this paper would be to numerically approximate the trace formula in equation (6.4). One would need to find many of the periodic orbits of the stadium billiard and compute the linear stability matrices for these. Then compare the statistics of these eigenvalues with those of random matrices.

8.1. **Question from Presentations.** Aalok asked a very intelligent question at the end of my talk on 15 May 2009. He asked, how do we know that the stadium billiard is a strongly chaotic system? There was discussion about the answer to this question being reduced to showing that the set of periodic orbits is a set of measure zero. This is not quite true. One must show that almost every initial condition comes arbitrarily close to any point in the phase space. This is a nontrivial calculation for the stadium billiard and is beyond the scope of this paper.

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