DECAY OF SINGLET CONVERSION PROBABILITY IN ONE DIMENSIONAL QUANTUM NETWORKS

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ABSTRACT. Quantum networks are used to transmit and process information by using the phenomena of quantum mechanics. Quantum states are prepared in some way that can be repeated and neighboring nodes of the network can be entangled to improve quantum information processing. Protocols are designed to optimize the probability that two distant nodes in the network are maximally entangled [5]. A protocol called entanglement swapping, was introduced to two dimensional quantum networks with much success [1]. By using measurements described by projective operators onto the Bell basis, it was shown that in large one dimensional networks the probability of entangling and converting two distant nodes into a singlet exponentially decreases with the number of stations [5]. In this paper, we introduce notation and a bit of linear algebra to set up the frame work of quantum mechanics in the setting of quantum information and quantum computation. Then we describe the protocol of entanglement swapping with a explicit example using the Bell basis. Next we consider entanglement swapping using an arbitrary basis and show decaying bounds of the singlet conversion probability (SCP).

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Contents

1. Introduction

Quantum networks are made of a series of nodes that can transmit information. The key to transmitting information is to have entangled pairs of nodes in a state called a *singlet*. If a pair of nodes is maximally entangled then the state is a singlet, however, if the nodes are in a different state and presumable separated by some nontrivial distance then we wish to find local measurements, which may be used with classical communication, to convert the state into a singlet. The probability that this can be done is called the *Singlet Conversion Probability* or SCP.

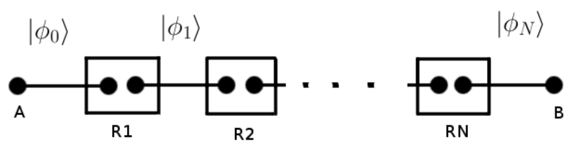

FIGURE 1. A quantum network of $N+1$ entangled pairs and N repeaters.

In a quantum network, an entangled pair is separated and the separate nodes are placed "close" to other nodes at a site also called a *repeater*. Local measurements may be made at the sites on any number of nodes with the goal to maximize the probability of obtaining a singlet between different nodes. As the number of repeaters increases, the probability decays, however, if the decay is polynomial, then the network is very efficient, if the decay is exponential then the network is useless.

A quantum protocol at the repeaters called *entanglement swapping* was introduced in [1] that drastically improved the SCP in the one repeater case and improved the overall efficiency of two dimensional networks. In large one dimensional quantum networks, this protocol, using the Bell basis, was found to be inefficient. In other words, the SCP of two distant nodes in a one dimensional network decays exponentially [5]. This was thought to be true with arbitrary measurements but was not proven rigorously. While the scope of this paper is not to prove exponentially decay, it is the author's intent to use the lemma proved in section (5) to prove exponential decay.

In this paper, we first set up the linear algebra notation that is familiar in quantum information and computation. Then we state the postulates of quantum mechanics as in [4]. Then we introduce the density operator language of quantum mechanics. This formulation of the postulates is equivalent to the previously stated postulates and the density operator allows one to analyze subsystems of joint quantum states. Next the entanglement swapping protocol is shown in an explicit example with the Bell basis and in the next section the entanglement swapping is carried out with an arbitrary measurement. Then we prove a proposition about the average singlet conversion probability in this context.

2. Quantum Information

2.1. **Hilbert Spaces.**

2.1.1. *Notation.* We first introduce common notation in the field of quantum information and quantum computing.

Definition 2.1. For a vector space V, elements or column vectors are denoted as $|v\rangle \in V$ and are called *kets*.

In quantum computing, the most common vector space is $\mathbb{C}^2$. So for $|v\rangle \in \mathbb{C}^2$ it may help to think of elements as

$$|v\rangle = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

for $v_i \in \mathbb{C}$. The row vectors in V are denoted by $\langle v|$ and called the bra v. This is convienent notation when we discuss the inner product. When we have the inner product of $v, w \in V$, often seen in mathematics as v^*w, the bra and ket notation gives, $v^*w = \langle v|w\rangle$, thus the inner product is a bracket. Every Hilbert space has a norm inducing inner product.

Definition 2.2. We define an *inner product* between two elements $|v\rangle, |w\rangle \in V$ as $(|v\rangle, |w\rangle) = \langle v|w\rangle$, with the properties

$$\begin{aligned} &1) \quad \langle v|w\rangle = \overline{\langle w|v\rangle} \\ &2) \quad \langle v|\lambda w\rangle = \lambda\langle v|w\rangle, \text{ for all } \lambda \in \mathbb{C}. \\ &3) \quad \langle v|v\rangle \geq 0, \text{ for all } v \in V, \langle v|v\rangle = 0 \text{ only if } v = 0. \end{aligned}$$

Another key ingredient in Hilbert spaces are linear operators. Since we will be working with finite dimensional Hilbert spaces, namely $\mathbb{C}^n$, all linear operators are represented by matrices. We say that an n by m matrix A with complex entries is an element of $\mathbb{C}^{n\times m}$ or $A \in \mathbb{C}^{n\times m}$. We denote the adjoint of a matrix $A \in \mathbb{C}^{n\times m}$ as

$$A^\dagger = \overline{(A^T)},$$

where the T is the transpose and $\bar{\cdot}$ is the complex conjugation of the entries of A^T. There are many special matrices also known as gates in quantum computing. We introduce a few and use them later in this paper.

Definition 2.3. The two by two *Pauli Matrices* are defined as

$$\sigma_0 = \sigma_I = I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \sigma_1 = \sigma_x = X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$\sigma_2 = \sigma_y = Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_3 = \sigma_z = Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Definition 2.4. The *Hadamard matrix (gate)* is given by the matrix

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{2.5}$$

Example 2.6. Consider the Hilbert Space $\mathbb{C}^2$ with the usual inner product. Let $\{|0\rangle, |1\rangle\}$ be any two dimensional orthonormal basis for $\mathbb{C}^2$. It is clear that the σ_0 Pauli matrix leaves any element it acts on it undisturbed. However, the other gates perform simple operations on the orthonormal basis.

$$\begin{aligned} \sigma_1|0\rangle &= |1\rangle & \sigma_1|1\rangle &= |0\rangle \\ i\sigma_2|0\rangle &= -|1\rangle & -i\sigma_2|1\rangle &= -|0\rangle \\ -\sigma_3|0\rangle &= -|0\rangle & \sigma_3|1\rangle &= -|1\rangle \end{aligned}$$

So the first Pauli matrix, σ_1, is a not gate. It sends 0 to 1 and 1 to 0. Similarly, the second Pauli matrix is a negative not gate and the third is a negative identity gate.

The Hadamard gate's usefulness is described later in section (2.3). When working with several closed quantum systems it is imperative to see these many systems as one big quantum system. To do so we need the machinery of tensor products.

Definition 2.7. Given V, W Hilbert spaces, we can define a *tensor* $|v\rangle \otimes |w\rangle = |vw\rangle$ for all $v \in V$, $w \in W$ as

1) $z(|v\rangle \otimes |w\rangle) = (z|v\rangle) \otimes |w\rangle = |v\rangle \otimes z|w\rangle$ for all $z \in \mathbb{C}$
2) $|v_1\rangle, |v_2\rangle \in V, |w\rangle \in W, \quad (|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1 w\rangle + |v_2 w\rangle.$
3) $|v\rangle \in V, |w_1\rangle, |w_2\rangle \in W, \quad |v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |vw_1\rangle + |vw_2\rangle.$

With this information, we now describe some elementary ideas in quantum mechanics knowing that our goal is to work with the most basic quantum system, the qubit.

2.2. **Postulates of Quantum Mechanics.** Quantum mechanics is a mathematical framework to develop physical theories. The postulates of quantum mechanics connect the mathematical framework to the physical world.

Postulate 2.8. *Associated to any isolated physical system is a complex vector space with an inner product called the state space. The system is described by the state vector.*

Classical computers that we use everyday, have bits and these bits can be in one of two positions, a 0 or 1. In quantum computation, the classical bit is replaces by a qubit.

Definition 2.9. A *qubit* is the simplest quantum mechanical system and has a two dimensional state space. That is, the qubit can be modeled by the vector space $\mathbb{C}^2$ with some arbitrary orthonormal basis $\{|0\rangle, |1\rangle\}$.

Postulate 2.10. *The evolution of a closed (i.e. not interacting with any other system) quantum system is described by a unitary operation U. For a system in state $|\psi\rangle = |\psi(t)\rangle$ at time t, at a later time $t+s$ it follows*

$$|\psi(t+s)\rangle = U|\psi(t)\rangle.$$

For example, the quantum not gate described by taking a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ to $|\psi'\rangle = \alpha|1\rangle + \beta|0\rangle$ is described by the Pauli matrix σ_1. That is $|\psi'\rangle = \sigma_2|\psi\rangle$.

Postulate 2.11. *Let $\{M_m\}$ be measurement operations. The $\{M_m\}$ must satisfy*

$$\sum_m M_m^\dagger M_m = I. \tag{2.12}$$

If the quantum system is in state $|\psi\rangle$ immediately before the measurement is taken, then the probability that the result of the measurement is m is given by

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle,$$

and the state of the system immediately after the measurement is given by

$$\frac{M_m|\psi\rangle}{\sqrt{p(m)}}.$$

Notice that equation (2.12), equates to the probabilities summing to one.

Postulate 2.13. *The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. For example, if the n systems are described by the states $|\psi_1\rangle, ..., |\psi_n\rangle$, then the joint state of the total system is*

$$|\Psi\rangle = |\psi_1\rangle \otimes ... \otimes |\psi_n\rangle.$$

A joint system consisting of two qubits can be thought of in two ways. One is a vector $|v\rangle \in \mathbb{C}^2$ describing the first qubit, tensored with $|w\rangle \in \mathbb{C}^2$ describing the second qubit. Another way is to describe the entire two qubit system with one vector $|V\rangle \in \mathbb{C}^4$. Thus a two qubit system has a state space in $\mathbb{C}^4$ and thus we have a four dimensional orthonormal basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. An important basis for a two qubit system in the Bell basis, also know as the Bell states.

Example 2.14. The Bell states make up an orthonormal basis in $\mathbb{C}^4$ and can be used to form projective measurements on quantum systems.

Definition 2.15. The *Bell states*, $|\Phi^\pm\rangle, |\Psi^\pm\rangle$, are defined as

$$|\Phi^\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \qquad |\Psi^\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}.$$

Definition 2.16. The Bell measurements are given by $\{\Phi^\pm, \Psi^\pm\}$,

$$\Phi^\pm = |\Phi^\pm\rangle\langle\Phi^\pm|, \qquad \Psi^\pm = |\Psi^\pm\rangle\langle\Psi^\pm|.$$

First we show that the Bell measurements satisfy the completeness equation, that is

$$(\Phi^+)^\dagger\Phi^+ + (\Phi^-)^\dagger\Phi^- + (\Psi^+)^\dagger\Psi^+ + (\Psi^-)^\dagger\Phi^- = I.$$

Proof. Consider Φ^+. Then

$$(\Phi^+)^\dagger\Phi^+ = (|\Phi^+\rangle\langle\Phi^+|)^\dagger|\Phi^+\rangle\langle\Phi^+|.$$

Since all the bell states are purely real, then $(\Phi^+)^\dagger = \Phi^+$ and thus

$$(\Phi^+)^\dagger\Phi^+ = |\Phi^+\rangle\langle\Phi^+|\Phi^+\rangle\langle\Phi^+| = |\Phi^+\rangle\langle\Phi^+| = \Phi^+,$$

since the Bell states are orthonormal. Thus

$$(\Phi^+)^\dagger\Phi^+ + (\Phi^-)^\dagger\Phi^- + (\Psi^+)^\dagger\Psi^+ + (\Psi^-)^\dagger\Phi^- = \Phi^+ + \Phi^- + \Psi^+ + \Psi^-.$$

The Bell states form an orthonormal basis for $\mathbb{C}^4$ and by the completion relation,

$$\Phi^+ + \Phi^- + \Psi^+ + \Psi^- = I,$$

and this completes the proof.

□

Therefore the bell measurements satisfy the completeness equation and can be used to measure quantum states. In section (4.1), we use the bell measurements as a protocol in the transmission of quantum information.

2.3. **Quantum Information.** An interesting quantum problem is given two parties, who share an entangled pair of quibits, how does one party communicate the state of an unknown third qubit? Suppose Alice and Bob prepare a Bell state between two qubits, namely $|\Phi^+\rangle$. Then Alice and Bob take their qubits and move far away. Next, Alice is given a third qubit in some unknown state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Now Alice must use her entangled qubit and classical communication so that Bob will be able to reproduce the state $|\psi\rangle$. First the joint state of the entire three qubit system is

$$|\psi_0\rangle = |\psi\rangle \otimes |\Phi^+\rangle = \alpha|0\rangle \otimes \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) + \beta|1\rangle \otimes \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right).$$

Alice is in posession of the first two qubits of this system. First Alice can send her two qubits through the controlled not gate (CNOT). In short this gate checks the first qubit. If the qubit is a $|0\rangle$ then the second qubit is left unchanged. If the first qubit is a $|1\rangle$, then the second qubit is flipped regardless of the initial state. Thus the resulting state after the measurement is

$$\psi_1\rangle = \alpha|0\rangle \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) + \beta|1\rangle \left(\frac{|10\rangle + |01\rangle}{\sqrt{2}}\right).$$

Next Alice sends her first qubit through the Hadamard gate which sends

$$|0\rangle \to \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \tag{2.17}$$

$$|1\rangle \to \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \tag{2.18}$$

This gives the state

$$|\psi_2\rangle = \frac{1}{2}(\alpha(|0\rangle + |1\rangle) \otimes (|00\rangle + |11\rangle) + \beta(|0\rangle - |1\rangle) \otimes (|10\rangle + |01\rangle).$$

Next, we write this system as the two qubits Alice has tensored with Bob's qubit.

$$|\psi_2\rangle = \frac{1}{2}(|00\rangle\otimes(\alpha|0\rangle+\beta|1\rangle)+|01\rangle\otimes(\alpha|1\rangle+\beta|0\rangle)+|10\rangle\otimes(\alpha|0\rangle-\beta|1\rangle)+|11\rangle\otimes(\alpha|1\rangle-\beta|0\rangle)).$$

The benefit of this notation is that it's clear to see that Alice can perform the projective measurements of the four basis elements in $\mathbb{C}^2 \otimes \mathbb{C}^2$ and if Alice gets the result 00, then Bob's qubit is in state $\alpha|0\rangle + \beta|1\rangle$ which is the state $|\psi\rangle$. With these results, Alice can classically communicate her result to Bob. Then Bob must use the identity gate I for a result of 00, X gate for 01, Z gate for 10, and X then Z gate for 11. Bob's qubit then will be in state $|\psi\rangle$.

3. Density operators

Thus far, quantum mechanical systems have been completely described by state vectors in some state space. It will be useful later to describe a system by a density matrix also known as a density operator. First we define the density operator within the example of an ensemble of states.

Definition 3.1. Suppose a quantum system is in one of a number of state $|\psi_i\rangle$ with probability p_i. Then $\{|\psi_i\rangle, p_i\}$ is an *ensemble of pure states*. Furthermore, the *density operator* is defined as

$$\rho \equiv \sum_i p_i|\psi_i\rangle\langle\psi_i|.$$

3.1. **The Postulates of Quantum Mechanics using the Density Operator.** From this formulation of describing a state using an operator, we can reformulate the postulates of quantum mechanics. This reformulation is equivalent to the postulates introduced in section (2.2). We briefly state these postulates in the density language.

Postulate 3.2. *Associated to any isolated physical system is a complex vector space with an innerproduct called the state space. The system is completely described by a density operator* ρ*. The density operator is a positive operator with trace one acting on the state space of the system. If the quantum system is in state* ρ_i *with probability* p_i*, then the density operator for the system is* $\sum_i p_i\rho_i$.

Postulate 3.3. *The evolution of a closed quantum system is described by a unitary transformation* U*. For a system in state* $\rho = \rho(t)$ *at time* t*, at a later time* $t+s$ *it follows*

$$\rho(t+s) = U\rho(t)U^\dagger.$$

Postulate 3.4. *Given a quantum measurement described by the measurement operators* $\{M_m\}$*, and a quantum system in the state* ρ *immediately before the measurement, then the probability that result* m *occurs is given by*

$$p(m) = tr(M_m^\dagger M_M \rho),$$

and the state of the system after the measurement is

$$\frac{M_m \rho M_m^\dagger}{p(m)}.$$

Postulate 3.5. *The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. For example, if the* n *systems are labeled* $\rho_1, ..., \rho_n$*, then the joint state of the total system is*

$$\rho = \rho_1 \otimes ... \otimes \rho_n.$$

3.2. **Reduced Density Operator.** The density operator is an important tool when studying a subsystem of a composite quantum system. To analyse the subsystem of a composite system we introduce the reduced density operator.

Definition 3.6. Suppose there are systems A and B in a joint state is described by the density operator ρ^{AB}. The *reduced density operator* for system A is

$$\rho^A \equiv \mathrm{tr}_B(\rho^{AB}),$$

where tr_B is the partial trace over the system B.

Definition 3.7. The *partial trace* is defined as

$$\mathrm{tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) \equiv |a_1\rangle\langle a_2|\mathrm{tr}(|b_1\rangle\langle b_2|) = |a_1\rangle\langle a_2|\langle b_2|b_1\rangle,$$

for any $|a_1\rangle, |a_2\rangle \in A$ and $|b_1\rangle, |b_2\rangle \in B$.

Thus, given a density, $\rho^{1,\dots,n}$, of a joint system with state spaces $A_1, A_2, ..., A_n$, we can find the density of any subsystem by using the partial trace operator. For example, the denisty operator of the joint states A_j and A_k, denoted $\rho^{j,k}$ is

$$\rho^{j,k} = tr_{1,2,\dots,j-1,j+1,\dots,k-1,k+1,\dots,n}(\rho^{1,\dots,n}).$$

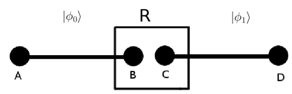

FIGURE 2. A one repeater situation in which $|\phi_0\rangle = \phi_1\rangle$.

4. A SYSTEM OF QUBITS WITH REPEATERS

4.1. **Entanglement swapping.**

Consider the case in figure (2), which two qubits are prepared in state $|\phi_0\rangle \in \mathbb{C}^2$ then separated. Then two other qubits are prepared in the same state $|\phi_0\rangle = |\phi_1\rangle \in \mathbb{C}^2$ then separated so that one of the qubits is "close" to one of the qubits from the first state. These two "close" qubits are called a repeater. By the Schmidt decomposition theorem (in the appendix, section (7)), we can write

$$|\phi_0\rangle = |\phi_1\rangle = \sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle.$$

By the fourth postulate of quantum mechanics, the state of the entire four qubit system can be written as

$$\begin{aligned}|\phi_1\rangle \otimes |\phi_1\rangle &= (\sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle) \otimes (\sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle)\\ &= \lambda_0|0000\rangle + \sqrt{\lambda_0\lambda_1}|0011\rangle + \sqrt{\lambda_0\lambda_1}|1100\rangle + \lambda_1|1111\rangle.\end{aligned}$$

Now we wish to perform the Bell measurements on the repeater, and leave the other two qubits undisturbed. Thus, in the four qubit system, for the first Bell measurement Φ^+ we make the measurement $I_1 \otimes (\Phi^+)_{23} \otimes I_4$. Where the subscripts denote which qubits the operator is acting on. So the resulting state of the measurement Φ^+ is

$$\begin{aligned}\frac{I \otimes |\Phi^+\rangle\langle\Phi^+| \otimes I|\phi\rangle}{p(\Phi^+)^{1/2}} &= p(\Phi^+)^{-1/2}\left(\lambda_1|0\rangle \otimes \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right)|00\rangle \otimes |0\rangle\right.\\ &+\sqrt{\lambda_1\lambda_2}|0\rangle \otimes \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right)|01\rangle \otimes |1\rangle\\ &+\sqrt{\lambda_1\lambda_2}|1\rangle \otimes \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right)|10\rangle \otimes |0\rangle\\ &\left.+\lambda_2|1\rangle \otimes \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right)|11\rangle \otimes |1\rangle\right)\end{aligned}$$

Where $p(\Phi^+)$ is the probability of the resulting state which will be computed soon. The inner products give $\langle ii|ii\rangle = 1$ and all the other inner products are zero. This gives the result

$$\begin{aligned}\frac{I \otimes |\Phi^+\rangle\langle\Phi^+| \otimes I|\phi\rangle}{p(\Phi^+)^{1/2}} &= p(\Phi^+)^{-1/2}\left(\lambda_1|0\rangle \otimes \frac{1}{2}(|00\rangle + |11\rangle) \otimes |0\rangle + \lambda_2|1\rangle \otimes \frac{1}{2}(|00\rangle + |11\rangle) \otimes |1\rangle\right)\\ &= p(\Phi^+)^{-1/2}\left(\frac{\lambda_1}{\sqrt{2}}(|0\rangle \otimes |\Phi^+\rangle \otimes |0\rangle) + \frac{\lambda_2}{\sqrt{2}}(|1\rangle \otimes |\Phi^+\rangle \otimes |1\rangle)\right).\end{aligned}$$

With probability

$$p(\Phi^+) = \langle\phi|(I \otimes |\Phi^+\rangle\langle\Phi^+| \otimes I|)^\dagger(I \otimes |\Phi^+\rangle\langle\Phi^+| \otimes I|)|\phi\rangle = \frac{\lambda_1^2 + \lambda_2^2}{2}.$$

For Φ^-, not much changes due to the minus,

$$\frac{I\otimes|\Phi^-\rangle\langle\Phi^-|\otimes I|\phi\rangle}{p(\Phi^-)^{1/2}} = p(\Phi^-)^{-1/2}\left(\lambda_1|0\rangle\otimes\frac{1}{2}(|00\rangle-|11\rangle)\otimes|0\rangle+\lambda_2|1\rangle\otimes\frac{-1}{2}(|00\rangle-|11\rangle)\otimes|1\rangle\right)$$
$$= p(\Phi^-)^{-1/2}\left(\frac{\lambda_1}{\sqrt{2}}(|0\rangle\otimes|\Phi^-\rangle\otimes|0\rangle)-\frac{\lambda_2}{\sqrt{2}}(|1\rangle\otimes|\Phi^-\rangle\otimes|1\rangle)\right).$$

With probability the same as Φ^+. Now for $\Psi^\pm$ we have to use the other terms. So the state after the measurement Ψ^+ is

$$\frac{I\otimes|\Psi^+\rangle\langle\Psi^+|\otimes I|\phi\rangle}{p(\Psi^+)^{1/2}} = p(\Psi^+)^{-1/2}\left(\lambda_1|0\rangle\otimes\left(\frac{|01\rangle+|10\rangle}{\sqrt{2}}\right)\left(\frac{\langle 01|+\langle 10|}{\sqrt{2}}\right)|00\rangle\otimes|0\rangle\right.$$
$$+\sqrt{\lambda_1\lambda_2}|0\rangle\otimes\left(\frac{|01\rangle+|10\rangle}{\sqrt{2}}\right)\left(\frac{\langle 01|+\langle 10|}{\sqrt{2}}\right)|01\rangle\otimes|1\rangle$$
$$+\sqrt{\lambda_1\lambda_2}|1\rangle\otimes\left(\frac{|01\rangle+|10\rangle}{\sqrt{2}}\right)\left(\frac{\langle 01|+\langle 10|}{\sqrt{2}}\right)|10\rangle\otimes|0\rangle$$
$$\left.+\lambda_2|1\rangle\otimes\left(\frac{|01\rangle+|10\rangle}{\sqrt{2}}\right)\left(\frac{\langle 01|+\langle 10|}{\sqrt{2}}\right)|11\rangle\otimes|1\rangle\right)$$
$$= p(\Psi^+)^{-1/2}(\frac{\sqrt{\lambda_1\lambda_2}}{2}|0\rangle\otimes(|01\rangle+|10\rangle)\otimes|1\rangle+\frac{\sqrt{\lambda_1\lambda_2}}{2}|1\rangle\otimes(|01\rangle+|10\rangle)\otimes|0\rangle$$
$$= p(\Psi^+)^{-1/2}\frac{\sqrt{\lambda_1\lambda_2}}{\sqrt{2}}(|0\rangle\otimes|\Psi^+\rangle\otimes|1\rangle+|1\rangle\otimes|\Psi^+\rangle\otimes|0\rangle).$$

With probability

$$p(\Psi^+) = \langle\phi|(I\otimes|\Psi^+\rangle\langle\Psi^+|\otimes I|)^\dagger(I\otimes|\Psi^+\rangle\langle\Psi^+|\otimes I|)|\phi\rangle = \lambda_1\lambda_2.$$

For the measurement Ψ^-, the probability does not change from Ψ^+ and the resulting state is

$$\frac{I\otimes|\Psi^-\rangle\langle\Psi^-|\otimes I|\phi\rangle}{p(\Psi^-)^{1/2}} = p(\Psi^-)^{-1/2}\frac{\sqrt{\lambda_1\lambda_2}}{\sqrt{2}}(|0\rangle\otimes|\Psi^-\rangle\otimes|1\rangle-|1\rangle\otimes|\Psi^-\rangle\otimes|0\rangle).$$

Next we are interested in the resulting state between the two end qubits. To do so we use the partial trace operator. So the density of the resulting state of $\Phi^\pm$ is

$$\rho = p(\Phi^\pm)^{-1}\left(\frac{\lambda_1}{\sqrt{2}}(|0\rangle\otimes|\Phi^\pm\rangle\otimes|0\rangle)+\frac{\lambda_2}{\sqrt{2}}(|1\rangle\otimes|\Phi^\pm\rangle\otimes|1\rangle\right)\left(\frac{\lambda_1}{\sqrt{2}}(\langle 0|\otimes\langle\Phi^\pm|\otimes\langle 0|)+\frac{\lambda_2}{\sqrt{2}}(\langle 1|\otimes\langle\Phi^\pm|\otimes\langle 1|\right).$$

We trace out the middle two qubits using the partial trace to get

$$\rho' = |\phi^\pm\rangle\langle\phi^\pm|,$$

where

$$|\phi^\pm\rangle = \frac{\lambda_1|00\rangle\pm\lambda_2|11\rangle}{\sqrt{\lambda_1^2+\lambda_2^2}},$$

with probability

$$\frac{\lambda_1^2+\lambda_2^2}{2},$$

Like wise

$$|\psi^\pm\rangle = \frac{|01\rangle\pm|10\rangle}{\sqrt{2}},$$

with probability $\lambda_1\lambda_2$.

4.2. **General Measurments between two entangled states.** Consider a one repeater system with qubits at points A, B, C, D with AB in state

$$|\phi_1\rangle = \sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle$$

and CD in state

$$|\phi_2\rangle = \sqrt{\mu_0}|00\rangle + \sqrt{\mu_1}|11\rangle$$

Then the joint state of $ABCD$ is

$$|\phi_1\rangle \otimes |\phi_2\rangle = \sqrt{\lambda_0\mu_0}|0000\rangle + \sqrt{\lambda_0\mu_1}|0011\rangle + \sqrt{\lambda_1\mu_0}|1100\rangle + \sqrt{\lambda_1\mu_1}|1111\rangle.$$

Next we wish to consider the projective measurement $|u_m\rangle\langle u_m|$ on the one repeater. The basis kets $\{|ij\rangle,\ i = 0, 1,\ j = 0, 1\}$, where the basis is from the BC qubits, form an orthonormal basis in $\mathbb{C}^4$. Thus any projective measurement is going to be a sum of the basis kets in $\mathbb{C}^4$. So we may write $|u_m\rangle$ as

$$|u_m\rangle = \sum_{i,j=0}^{1} a_{i,j,m}|ij\rangle,$$

with

$$\sum_{i,j=0}^{1} |a_{i,j,m}|^2 = 1.$$

The $|u_m\rangle$ are a basis in $\mathbb{C}^4$ so,

$$\sum_{m=1}^{4} |u_m\rangle\langle u_m| = I.$$

The $|u_m\rangle$ are orthonormal, so $\langle u_m|u_n\rangle = 0$ if $n \neq m$ and 1 if $n = m$. To aid in our computations using the general measurement operators $|u_m\rangle\langle u_m|$, we carry out a quick calculation using the basis kets $\{|ij\rangle,\ i = 0, 1,\ j = 0, 1\}$ as projective measurments $|ij\rangle\langle ij|$, and we get the follwing probabilities,

$$\begin{aligned}
p(00) =&\langle \phi_1 \otimes \phi_2|I_1 \otimes (|00\rangle\langle 00|)_{23} \otimes I_4|\phi_1 \otimes \phi_2\rangle = |\langle(\phi_1 \otimes \phi_2)_{23}|00\rangle|^2 = \lambda_0\mu_0,\\
p(11) =&\langle \phi_1 \otimes \phi_2|I_1 \otimes (|11\rangle\langle 11|)_{23} \otimes I_4|\phi_1 \otimes \phi_2\rangle = |\langle(\phi_1 \otimes \phi_2)_{23}|11\rangle|^2 = \lambda_1\mu_1,\\
p(10) =&\langle \phi_1 \otimes \phi_2|I_1 \otimes (|10\rangle\langle 01|)_{23} \otimes I_4|\phi_1 \otimes \phi_2\rangle = |\langle(\phi_1 \otimes \phi_2)_{23}|10\rangle|^2 = \lambda_1\mu_0,\\
p(01) =&\langle \phi_1 \otimes \phi_2|I_1 \otimes (|01\rangle\langle 10|)_{23} \otimes I_4|\phi_1 \otimes \phi_2\rangle = |\langle(\phi_1 \otimes \phi_2)_{23}|01\rangle|^2 = \lambda_0\mu_2,
\end{aligned}$$

where the subscripts on the operators denote which qubit(s) the operators are acting on. Next, consider the measurements done with the orthonormal basis $\{|u_m\rangle, m = 1, 2, 3, 4\}$. Then the resulting probability $p(u_m)$ of one of these outcomes would be

$$p(u_m) = \langle \phi_1 \otimes \phi_2|(|u_m\rangle\langle u_m|)|\phi_1 \otimes \phi_2\rangle = \sum_{i,j=0}^{1} |a_{i,j,m}\langle(\phi_1 \otimes \phi_2)_{23}|ij\rangle|^2,$$

or,

$$p(u_m) = |a_{0,0,m}|^2\lambda_0\mu_0 + |a_{1,1,m}|^2\lambda_1\mu_1 + |a_{1,0,m}|^2\lambda_1\mu_0 + |a_{0,1,m}|^2\lambda_0\mu_1.$$

The resulting state would be

$$
\begin{aligned}
|\psi\rangle =& \frac{I_1 \otimes (|u_m\rangle\langle u_m|)_{23} \otimes I_4 |\phi_1 \otimes \phi_2\rangle}{\sqrt{p(u_m)}} \\
=& \left(\frac{1}{\sqrt{p(u_m)}}\right) (a_{0,0,m}\sqrt{\lambda_0\mu_0}|0\rangle \otimes |u_m\rangle \otimes |0\rangle + a_{0,1,m}\sqrt{\lambda_0\mu_1}|0\rangle \otimes |u_m\rangle \otimes |1\rangle \\
&+ a_{1,0,m}\sqrt{\lambda_1\mu_0}|1\rangle \otimes |u_m\rangle \otimes |0\rangle + a_{1,1,m}\sqrt{\lambda_1\mu_1}|1\rangle \otimes |u_m\rangle \otimes |1\rangle),
\end{aligned} \tag{4.1}
$$

We wish to analyze the state between qubits AD, that is we need information on the two qubit subsystem of the four qubit composite state we know in equation (4.1). To do so, we refer to the density operator of the system as described in section (3). The density of the joint state between qubits $ABCD$ after the measurement is

$$\rho^{ABCD} = |\psi\rangle\langle\psi|.$$

To find the density of the reduced state between qubits AD, we trace out the second and third qubits by using the partial trace. This results in

$$
\begin{aligned}
\rho^{AD} = \mathrm{tr}_{BC}(\rho^{ABCD}) = &\left(\frac{\mathrm{tr}(|u_m\rangle\langle u_m|)}{\sqrt{p(u_m)}}\right) \Big[a_{0,0,m}\sqrt{\lambda_0\mu_0}|00\rangle(\langle 00|\overline{a_{0,0,m}}\sqrt{\lambda_0\mu_0} \\
&+ \langle 10|\overline{a_{0,1,m}}\sqrt{\lambda_0\mu_1} + \langle 01|\overline{a_{1,0,m}}\sqrt{\lambda_1\mu_0} + \langle 11|\overline{a_{1,1,m}}\sqrt{\lambda_1\mu_1}) \\
&+ a_{0,1,m}\sqrt{\lambda_0\mu_1}|01\rangle(\langle 00|\overline{a_{0,0,m}}\sqrt{\lambda_0\mu_0} + \langle 10|\overline{a_{0,1,m}}\sqrt{\lambda_0\mu_1} + \langle 01|\overline{a_{1,0,m}}\sqrt{\lambda_1\mu_0} + \langle 11|\overline{a_{1,1,m}}\sqrt{\lambda_1\mu_1}) \\
&+ a_{1,0,m}\sqrt{\lambda_1\mu_0}|10\rangle(\langle 00|\overline{a_{0,0,m}}\sqrt{\lambda_0\mu_0} + \langle 10|\overline{a_{0,1,m}}\sqrt{\lambda_0\mu_1} + \langle 01|\overline{a_{1,0,m}}\sqrt{\lambda_1\mu_0} + \langle 11|\overline{a_{1,1,m}}\sqrt{\lambda_1\mu_1}) \\
&+ a_{1,1,m}\sqrt{\lambda_1\mu_1}|11\rangle(\langle 00|\overline{a_{0,0,m}}\sqrt{\lambda_0\mu_0} + \langle 10|\overline{a_{0,1,m}}\sqrt{\lambda_0\mu_1} + \langle 01|\overline{a_{1,0,m}}\sqrt{\lambda_1\mu_0} + \langle 11|\overline{a_{1,1,m}}\sqrt{\lambda_1\mu_1})\Big].
\end{aligned}
$$

Since the $\mathrm{tr}(|u_m\rangle\langle u_m|) = \langle u_m|u_m\rangle = 1$, this is an outer product, $\rho^{AD} = |\psi_{AD}\rangle\langle\psi_{AD}|$ where,

$$
\begin{aligned}
|\psi_{AD}\rangle =& \frac{\sum_{i,j=0}^{1} |ij\rangle a_{i,j,m}\langle(\phi_1 \otimes \phi_2)_{2,3}|ij\rangle}{\sqrt{p(u_m)}}, \\
=& \left(\frac{1}{\sqrt{p(u_m)}}\right) \Big(a_{0,0,m}\sqrt{\lambda_0\mu_0}|00\rangle + a_{0,1,m}\sqrt{\lambda_0\mu_1}|01\rangle + a_{1,0,m}\sqrt{\lambda_0\mu_1}|10\rangle + a_{1,1,m}\sqrt{\lambda_1\mu_1}|11\rangle\Big).
\end{aligned}
$$

Thus the state between AD is a pure state described by the ket $|\psi_{AD}\rangle$. Now we can describe the resulting state between AD using the Schmidt decomposition theorem, which can be found with proof in the appendix. We can find a new basis such that

$$|\psi_{AD}\rangle = \sqrt{\lambda_0^{AD}}|00\rangle + \sqrt{\lambda_1^{AD}}|11\rangle,$$

with $1 \geq \lambda_0^{AD} \geq \lambda_1^{AD}$ and $\lambda_0^{AD} + \lambda_1^{AD} = 1$. Furthermore, from the proof of the Schmidt decomposition theorem, $\lambda_{0,1}^{AD}$ are the squares of the singular values of the matrix,

$$\frac{1}{\sqrt{p(u_m)}} A(\lambda, \mu, m) = \frac{1}{\sqrt{p(u_m)}} \begin{bmatrix} a_{0,0,m}\sqrt{\lambda_0\mu_0} & a_{0,1,m}\sqrt{\lambda_0\mu_1} \\ a_{1,0,m}\sqrt{\lambda_1\mu_0} & a_{1,1,m}\sqrt{\lambda_1\mu_1} \end{bmatrix}.$$

Definition 4.2. For any n by m matrix $A \in \mathbb{C}^{n\times m}$ define $s_i(A)$, $i = 0, 1, ..., n-1$, as the $(i+1)^{\mathrm{st}}$ singular value of A. By the properties of singular values, $s_0(A) \geq$

$s_1(A) \geq ... \geq s_{n-1}(A) \geq 0$. Also, the function s_i, for each $i = 0, 1, ..., n-1$, is positively homogeneous. That is, for all $a \in \mathbb{R}$, $a > 0$, we have

$$s_i(aA) = as_i(A).$$

Thus the resulting state between AD can be written as

$$|\psi_{AD}\rangle = \frac{s_0(A(\lambda, \mu, m))}{\sqrt{p(u_m)}}|00\rangle + \frac{s_1(A(\lambda, \mu, m))}{\sqrt{p(u_m)}}|11\rangle.$$

5. Measurement with an Ensemble of States

Consider the four qubit situation from before, with qubits A, B, C, and D. Let the state between AB be in an ensemble of states (λ_1^k, p_k). That is AB is in state

$$|\psi_{AB}^k\rangle = \sqrt{\lambda_0^k}|00\rangle + \sqrt{\lambda_1^k}|11\rangle,$$

with probability p_k. The state CD is in an ensemble of states (μ_1^ℓ, q_ℓ), that is

$$|\psi_{CD}^\ell\rangle = \sqrt{\mu_0^\ell}|00\rangle + \sqrt{\mu_1^\ell}|11\rangle,$$

with probability q_ℓ.

Definition 5.1. The *singlet conversion probability* (SCP) of a state $|\psi\rangle = \sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle$ is the probability that $|\psi\rangle$ can be transformed into a perfect single. This probibility is known to be $2\lambda_1$ [6].

Definition 5.2. For an ensemble of states described by $\{\lambda_1^k, p_k\}$ with smallest Schmidt coefficients of the states λ_1^k and probabilities p_k, the *Average SCP* is given by

$$\overline{\text{SCP}} = 2\sum_k p_k \lambda_1^k.$$

Next we prove a theorem about the decay of the average SCP after a measurement between an ensemble of states and a static state.

Proposition 5.3. *Let $\{|u_m\rangle\langle u_m|\}_{m=1}^4$ be projective measurements like the ones described in section (4.2). Consider the four qubit system with ensembles (λ_1^k, p_k) between AB and (μ_1^ℓ, q_ℓ) on CD. Then after an arbitrary projective measurement*

$$\overline{SCP}_{AD} \leq 2\sum_{k,\ell} p_k q_\ell \min\{\lambda_1^k, \mu_1^\ell\}. \tag{5.4}$$

Proof. For fixed k and ℓ, we compute the resulting states and probabilities of a measurement at BC. The state at AB is

$$|\psi_{AB}\rangle = \sqrt{\lambda_0^k}|00\rangle + \sqrt{\lambda_1^k}|11\rangle,$$

and the state at CD is

$$|\psi_{CD}\rangle = \sqrt{\mu_0^\ell}|00\rangle + \sqrt{\mu_1^\ell}|11\rangle.$$

For all measurements of the form $\{|u_m\rangle\langle u_m|\}_{m=1}^4$ on qubits BC, we can find the resulting state from equation (4.2) and write

$$|\psi_{AD}\rangle = \frac{s_0(A(\lambda^k, \mu^\ell, m))}{\sqrt{p(u_m)}}|00\rangle + \frac{s_1(A(\lambda^k, \mu^\ell, m))}{\sqrt{p(u_m)}}|11\rangle. \tag{5.5}$$

Thus after the measurement, AD is in the state $|\psi_{AD}\rangle$ with probability $p(u_m)$. This is true for fixed k, ℓ. To compute the average SCP between AD, we sum up the product of the smallest Schmidt coefficients with their respective probabilities. The smallest Schmidt coefficient is given as the square of the coefficient of $|11\rangle$ in equation (5.5). The probability that AD is in the state $|\psi_{AD}\rangle$ in equation (5.5) is the probability that AB is in the k^{th} state, p_k, times the probability that CD is in the ℓ^{th} state, q_ℓ, times the probability of the result u_m, $p(u_m)$, or $p_k q_\ell p(u_m)$. We now sum up over k, ℓ, and m to find the average SCP,

$$\overline{\text{SCP}}_{AD} = 2\sum_{k,\ell} p_k q_\ell \sum_{m=1}^{4} p(u_m)\left(\frac{s_1(A(\lambda^k,\mu^\ell,m))}{\sqrt{p(u_m)}}\right)^2 = 2\sum_{k,\ell} p_k q_\ell \sum_{m=1}^{4} s_1(A(\lambda^k,\mu^\ell,m))^2.$$

Now we have reduced the problem to estimating the square of the smallest singular value of the matrix

$$A(\lambda^k,\mu^\ell,m) = \begin{bmatrix} a_{0,0,m}\sqrt{\lambda_0^k\mu_0^\ell} & a_{0,1,m}\sqrt{\lambda_0^k\mu_1^\ell} \\ a_{1,0,m}\sqrt{\lambda_1^k\mu_0^\ell} & a_{1,1,m}\sqrt{\lambda_1^k\mu_1^\ell} \end{bmatrix}$$

By the construction of the singular value decomposition for an arbitrary two by two matrix $A \in \mathbb{C}^{2\times 2}$, we have the following property,

$$s_1(A)^2 = \min_{\|x\|_2=1, x\neq 0} \|Ax\|_2^2.$$

Consider the vector $x_0 = (0,1)^T$, with $\|x_0\|_2 = 1$. We have,

$$s_1(A(\lambda^k,\mu^\ell,m))^2 \leq \|A(\lambda^k,\mu^\ell,m)x_0\|_2^2 = \left\|\begin{bmatrix} a_{0,1,m}\sqrt{\lambda_0^k\mu_1^\ell} \\ a_{1,1,m}\sqrt{\lambda_1^k\mu_1^\ell}\end{bmatrix}\right\|_2^2 = |a_{0,1,m}|^2\lambda_0^k\mu_1^\ell + |a_{1,1,m}|^2\lambda_1^k\mu_1^\ell.$$

We use this estimate for all $m = 1, ..., 4$ to get

$$\overline{\text{SCP}}_{AD} \leq 2\sum_{k,\ell} p_k q_\ell \mu_1^\ell \left(\sum_{m=1}^{4} \lambda_0^k|a_{0,1,m}|^2 + \lambda_1^k|a_{1,1,m}|^2\right). \tag{5.6}$$

Lemma: For fixed i, j,

$$\sum_{m=1}^{4} |a_{i,j,m}|^2 = 1.$$

Proof. Define

$$|\mathbf{a_m}\rangle = \begin{bmatrix} a_{0,0,m} \\ a_{0,1,m} \\ a_{1,0,m} \\ a_{1,1,m} \end{bmatrix}.$$

The $|u_m\rangle$ are an orthonormal basis of $\mathbb{C}^4$ implies that $\{a_m\}$ are an orthonormal basis of $\mathbb{C}^4$. From these vectors we can build an orthogonal matrix,

$$\mathbf{A} = [|\mathbf{a_1}\rangle||\mathbf{a_2}\rangle||\mathbf{a_3}\rangle||\mathbf{a_4}\rangle].$$

Also there exists $\{|\mathbf{v_m}\rangle\} \in \mathbb{C}^{\mathbf{4}}$ such that

$$\mathbf{A} = \begin{bmatrix} \langle \mathbf{v_1}| \\ \hline \langle \mathbf{v_2}| \\ \hline \langle \mathbf{v_3}| \\ \hline \langle \mathbf{v_4}| \end{bmatrix} = \mathbf{V}^{\dagger}.$$

$\mathbf{A}$ is orthogonal which means, $\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{A}^{\dagger}\mathbf{A} = \mathbf{I}$. Using the $\mathbf{V}$ representation and that the adjoint of the adjoint is $\mathbf{V}$ gives us $\mathbf{V}^{\dagger}\mathbf{V} = \mathbf{I}$. Thus $\mathbf{v_m^{\dagger}v_n} = \mathbf{1}$ for all $m = n$ and 0 for all $m \neq n$. So the vectors $\{\mathbf{v_m}\}$ form an orthonormal basis of $\mathbb{C}^4$. Because of the construction of the matrix $\mathbf{V}$, the vectors $|\mathbf{v_m}\rangle = |\mathbf{v_{ij}}\rangle$ are

$$|\mathbf{v_{ij}}\rangle = \begin{bmatrix} \overline{a_{i,j,1}} \\ \overline{a_{i,j,2}} \\ \overline{a_{i,j,3}} \\ \overline{a_{i,j,4}} \end{bmatrix}.$$

Thus orthonormality of the $|\mathbf{v_{ij}}\rangle$ gives

$$\langle \mathbf{v_{ij}}|\mathbf{v_{ij}}\rangle = \sum_{\mathbf{m=1}}^{\mathbf{4}} |\mathbf{a_{i,j,m}}|^{\mathbf{2}} = \mathbf{1},$$

which proves the lemma. □

With this lemma we can simplify the estimate by summing over m in inequality (5.6) to obtain

$$\overline{\mathrm{SCP}}_{AD} \leq 2\sum_{k,\ell} p_k q_\ell \mu_1^\ell(\lambda_0^k + \lambda_1^k) = 2\sum_{k,\ell} p_k q_\ell \mu_1^\ell.$$

Now we estimate the average SCP with respect to μ_1^ℓ. To do so we must prove another lemma.
Lemma: For $A \in \mathbb{C}^{2\times 2}$,

$$s_1(A)^2 = s_1(A^{\dagger})^2.$$

Proof. The singular values of a matrix A are the positive square roots of the eigenvalues of $A^{\dagger}A$. Let γ be an eigenvalue of $A^{\dagger}A$ with eigenvector v_γ. Then

$$A^{\dagger}Av_\gamma = \gamma v_\gamma,$$
$$A(A^{\dagger}A)v_\gamma = \gamma A v_\gamma,$$
$$(A^{\dagger})^{\dagger}A^{\dagger}(Av_\gamma) = \gamma(Av_\gamma).$$

Thus γ is also an eigenvalue of $(A^{\dagger})^{\dagger}A^{\dagger}$ with eigenvector of Av_γ. The singular values of $(A^{\dagger})$ are the positive square roots of $(A^{\dagger})^{\dagger}A^{\dagger}$. Thus the singular values of A are also the singular values of $A^{\dagger}$. This proves the lemma. □

From this, consider $x_0 = (0,1)^T$. Then

$$\begin{aligned}
s_1(A(\lambda^k,\mu^\ell,m))^2 &= s_1(A^\dagger(\lambda^k,\mu^\ell,m))^2 \\
&\le \|A^\dagger(\lambda^k,\mu^\ell,m)^\dagger x_0\|_2^2 = \left\| \begin{bmatrix} \overline{a_{1,0,m}}\sqrt{\lambda_1^k\mu_0^\ell} \\ \overline{a_{1,1,m}}\sqrt{\lambda_1^k\mu_1^\ell} \end{bmatrix} \right\|_2^2 \\
&= |a_{1,0,m}|^2\lambda_1^k\mu_0^\ell + |a_{1,1,m}|^2\lambda_1^k\mu_1^\ell.
\end{aligned}$$

This is true for all k, m, thus

$$\begin{aligned}
\overline{\mathrm{SCP}}_{AD} &\le 2\sum_{k,\ell} p_k q_\ell \lambda_1^k \sum_{m=1}^{4} \mu_0^\ell |a_{1,0,m}|^2 + \mu_1^\ell |a_{1,1,m}|^2 \\
&= 2\sum_{k,\ell} p_k q_\ell \lambda_1^k(\mu_0^\ell + \mu_1^\ell) = 2\sum_{k,\ell} p_k q_\ell \lambda_1^k.
\end{aligned}$$

Thus

$$\overline{\mathrm{SCP}}_{AD} \le 2\sum_{k,\ell} p_k q_\ell \mu_1^\ell \quad \text{and} \quad \overline{\mathrm{SCP}}_{AD} \le 2\sum_{k,\ell} p_k q_\ell \lambda_1^k.$$

Therefore

$$\overline{\mathrm{SCP}}_{AD} \le 2\sum_{k,\ell} p_k q_\ell \min\{\lambda_1^k, \mu_1^\ell\}.$$

□

6. Conclusion

From this proposition, we see that the average SCP after any measurement is then lowered to the sum of the minimum Schmidt coefficients. Not proven in this paper, the Schmidt coefficient after a measurement gets split into 4 different Schmidt coefficients with 4 probabilities. at least one of these Schmidt coefficients will be bigger than the original and at least one will be smaller. Thus we see a splitting of Schmidt coefficients. This can be seen explicitly from the Bell basis example in section (4.1) we see that we have two Schmidt coefficients bigger than the original λ_1 and two smaller. From this splitting we can see that the mean SCP will decay to zero as the number of repeaters grows.

6.1. **Future Work.** To prove exponential decay more work is needed and will be pursued. A result would be considered a negative result, in the sense that the proof would show that one dimensional quantum networks with the entanglement swapping protocol would be inefficient. However, a rigorous proof would open resources to pursue other avenues of research in two and three dimensional networks. With two dimensional networks in which entanglement swapping has been shown to be extremely beneficial [1][5], different protocols can be investigated, as well as different geometrical properties of the two dimensional lattice.

7. Appendix

7.1. **Scmidt Decomposition Theorem.** Throughout the paper, we use the Scmidt Decomposition theorem to write the composite state of two qubits. The following states and proves the Schmidt decomposition theorem from [4].
Theorem: Suppose that $|\psi\rangle$ is a pure state of a composite system of qubits, AB. Then there exists orthonormal states $|i_A\rangle$ for qubit A and orthonormal states $|i_B\rangle$ for qubit B such that

$$|\psi\rangle = \sum_i \sqrt{\lambda_i}|i_A i_B\rangle = \sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle, \tag{7.1}$$

where $\sqrt{\lambda_i}$ are real numbers satisfying $\sum_i \lambda_i = 1$ and are called *Schmidt coefficients.*

Proof. Let, $|i\rangle$ and $|j\rangle$ be any fixed orthonormal basis for A and B respectively, $i = 0, 1$, $j = 0, 1$. Then

$$|\psi\rangle = \sum_{i,j} a_{i,j}|ij\rangle, \tag{7.2}$$

for some $a_{i,j} \in \mathbb{C}$. Define the two by two matrix $A = (A)_{n,m} = a_{i+1,j+1}$. Then by the singular value decomposition, there exists V, U, Σ such that

$$A = U\Sigma V^\dagger, \tag{7.3}$$

where U, V are unitary and Σ is diagonal with non-negative real elements. Now we write $|\psi\rangle$ using this singular value decompostion.

$$|\psi\rangle = \sum_{ijk}(U)_{ik}(\Sigma)_{kk}(V^\dagger)_{kj}|ij\rangle. \tag{7.4}$$

Now we can define

$$|i_A\rangle = \sum_j (U)_{ik}|i\rangle, \quad |i_B\rangle = \sum_j (V^\dagger)_{kj}|j\rangle, \tag{7.5}$$

and $\sqrt{\lambda_i} = (\Sigma)_{ii}$. Now,

$$|\psi\rangle = \sum_i \sqrt{\lambda_i}|i_A i_B\rangle. \tag{7.6}$$

It is easy to see that $|i_A\rangle$ and $|i_B\rangle$ are orthornomal since U and V are unitary. □

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