NUMERICAL APPROXIMATIONS ON NULL CONTROLLABILITY OF STRUCTURALLY DAMPED EQUATIONS

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A UNDERGRADUATE THESIS

Presented to the Faculty of

The Arts and Sciences College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Bachelor of Science

Major: Mathematics

Under the Supervision of George Avalos and Mike Gunderson

Lincoln, Nebraska

May, 2008

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The null controllability problem is considered for 2-D plates under hinged mechanical boundary conditions. The resulting partial differential equation (PDE) system referred to as a structurally damped equation, is considered to obtain optimal rates of blowup for the associated minimal energy function $\mathbb{E}_2^0(T)$, as terminal time $T \to 0$. The optimal blowup rate is $\mathcal{O}(T^{-\frac{3}{2}})$, which is abnormal considering null controllable PDEs generally have at least exponential blowup rates. It has been shown in [6], that finite dimensional spectral truncations of the PDE achieve the same rates. We set out to show that one achieves the same result by using the more complicated finite difference method to approximate the PDE.

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Chapter 1

Introduction and Statement of Results

1.1 Problem Statement

With H a Hilbert space, let S be a linear operator $S : D(S) \subset H \to H$ be positive and self adjoint. We consider the structurally damped wave equation

$$\begin{cases} w_{tt} + \rho S^{\alpha} w_t + Sw & = & u, \ t \in (0, T], \\ (w(0), w_t(0)) = (w_0, w_1) & \in & D(S) \times H \end{cases} \tag{1.1}$$

where the "control" $u(t)$ is a function in $L^2(0, T; H)$, and $\rho > 2$. ρ is a structural parameter pertaining to the characteristics of the plate. For this system we wish to analyze the *null controllability problem*. In general terms, we wish to find the control $u(t)$ such that the solution of (1.1) is taken from the initial state (w_0, w_1) to rest, $(0, 0)$ at terminal time T. By our system's parabolic behaviour we expect that it is null controllable for some arbitrarily short time $T > 0$. Therefore our definition of null controllability is:

Definition 1. *The system (1.1) is said to be null controllable, if for any terminal time* $T > 0$ *and arbitrary initial conditions* (w_0, w_1)*, there exists a control function* $u \in L^2(0, T; H)$ *such that the corresponding solution* (w, w_t) *to system (1.1) satisfies* $(w(T), w_t(T)) = (0, 0)$*.*

We wish to analyse what happens to the energy of the control when $T \to 0$. With $\alpha = 1/2$, it is shown in [6] that by using spectral truncations of the operator S the blowup rate of the control approaches $\mathcal{O}(T^{-\frac{3}{2}})$. In the last few years, it has been shown that such parabolic (or analytic) control systems exhibit rational rates of blowup for their minimal norm controls. This is in stark contrast to the exponential rates of blowup exhibited in many other PDE control settings (whether the control be exerted in the interior or boundary of the domain).

In this paper we wish to give a numerical way to solve the null controllability problem by using finite difference approximations for the operators S and S^α and then show that we can analyse this abstract PDE as an ordinary differential equation and that the energy of the control goes to $\mathcal{O}(T^{-\frac{3}{2}})$ as well.

First we define the *minimal energy function*:

Definition 2. *Let* $u^0(t; y_0)$*,* $0 \le t \le T$ *be the minimal* $L_2(0, T)$*-normed control taking arbitrary initial data* y_0 *to the origin at time* $t = T$*. Let the minimal norm,* $\mathbb{E}_2^0(T; y_0)$*, be defined as*

$$\mathbb{E}_2^0(T; y_0) = \left\{ \int_0^T \|u^0(t; y_0)\|_2^2 dt \right\}^{1/2},$$

and the minimal energy,

$$\mathbb{E}_2^0(T) = \sup_{\|y_0\|=1} \mathbb{E}_2^0(T; y_0).$$

it is clear that as $T \to 0$ our minimal energy will blowup. We wish to capture the precise rate in this paper. From [6] we have the following theorem:

Theorem 1.1.1. *For the system (1.1) the minimal energy of the control,* $\mathbb{E}_2^0$*, of* u *steering the initial state* $[w_0, w_1]$ *to* 0 *at time* $t = T$*, obeys the estimate*

$$\begin{aligned} \mathbb{E}_0^2(T) &= \sup_{\|x_0\|_2=1} \left\{ \int_0^T \|u^0(t;x_0)\|_2^2 dt \right\}^{\frac{1}{2}} \\ &= \mathcal{O}\left(T^{-\frac{3}{2}}\right), \quad \textit{as } T \to 0. \end{aligned}$$

There are infinitely many controls, u, that will satisfy our steering conditions. We want to know the dynamics of the minimal norm, $\mathbb{E}_2^0$, as terminal time T goes to 0. Kalman gives an explicit formula for finite dimensional systems, to compute the minimal $L_2(0,T)$-normed control, $u^0(T;x_0)$. This formula for the optimal control is computationally complex, and instead we shall use the findings of R. Triggianni in [7].

Theorem 1.1.2. *Consider the system*

$$\begin{cases} \dot{y} &= Ay + Bu, \ 0 < t \leq T, \\ y(0) &= y_0, \end{cases}$$

where A *is* $(n \times n)$*, and* B *is* $(n \times m)$*, with arbitrary initial condition* y_0*. Let* K *be the Kalman matrix where*

$$K = [B, AB, A^2B, ..., A^kB],$$

Where k *is the smallest positive integer such that* K *has full rank. That is,* *rank*$(K) = n$. *Let the* $(k+1)m \times 1$ *vector* $\nu(t)$ *be*

$$\nu(t) = \begin{bmatrix} \nu_0 \\ \nu_1 \\ \nu_2 \\ \vdots \\ \nu_k \end{bmatrix} = -K^{-1}e^{At}y_0 f_T(t),$$

where $f_T(t)$ *is a scalar real-valued function that satisfies the following conditions,*

- $f_T(0) = f_T(T) = f_T^{(1)}(0) = f_T^{(1)}(T) = ... = f_T^{(k-1)}(0) = f_T^{(k-1)}(T) = 0$
- $\int_0^T f_T(t)dt = 1$.

Then the $(m \times 1)$*-vector control* $u(t) = \nu_0 + \frac{d}{dt}(\nu_1) + \frac{d^2}{dt^2}(\nu_2) + ... + \frac{d^{(k-1)}}{dt^{(k-1)}}(\nu_k)$, *steers the arbitrary initial condition* y_0 *to* 0 *at time* $t = T$.

From the abstract system (1.1), we consider the case where $\alpha = \frac{1}{2}$ and $S^\alpha = S^{\frac{1}{2}} = A$, and hence $S = A^2$. Now we let $Aw = \Delta w$, the Laplacian operator, and likewise $A^2 w = \Delta^2 w$, the biharmonic operator. The objective of this paper is to now use numerical methods, namely the finite difference approximation, and the computational friendly control from R. Triggianni, mentioned above, to find the asymptotics of the control energy as $T \to 0$.

Now to model the movement of a vibrating plate, we introduce the following parabolic PDE with hinged boundary conditions:

$$\begin{cases} u &= w_{tt} + \rho\Delta w_t + \Delta^2 w, \ t \in [0,T], \ (x,y) \in \Omega, \\ \Delta w &= 0, \ w = 0, \ (x,y) \in \partial\Omega, \\ w(0) &= w_0, \ w_t(0) = w_1, \ (x,y) \in \Omega, \end{cases} \tag{1.2}$$

where Ω, a 2-D plate that we take to be the unit square, $\Omega = \{ (x, y) \mid 0 < x < 1,\ 0 < y < 1\}$ and $\partial\Omega$ is the boundary of the unit square. This equation models the movements of a vibrating plate with square root velocity dampening. Our boundary conditions $\Delta w = 0$ and $w = 0$ imply that the plate is held at rest along the boundary and that the second derivative is zero at the boundary, which implies at most a linear solution on the boundary. Hence they are 'hinged' boundary conditions, as they behave like the hinges of a door. u is the control function, and $\rho > 2$ is a constant containing structural information about the plate. We shall discuss later why we consider the case $\rho > 2$ and not $\rho \leq 2$.

Since the brunt of my work has been in computation we come to the question of how to numerically compute the operator Δw. So we consider Poisson's equation in two dimensions with homogeneous Dirichlet boundary conditions and a source function f.

$$\begin{cases} -\Delta w &= f, \text{ on } \Omega, \\ w &= 0, \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

The boundary conditions allow us to easily implement the finite difference method to approximate a solution to (1.3).

1.2 Finite Difference Method

Consider the system (1.3) on the domain Ω, with arbitrary Dirichlet boundary conditions on $\partial\Omega$. Place a grid on Ω by choosing an integer n and creating an $(n+2)$ by $(n+2)$ mesh with $(n+2)^2$ total nodes, $(n)^2$ interior nodes and $4n+4$ nodes on the boundary. There is a step size of $\frac{1}{n+1}$ between any two adjacent nodes. Each node is labeled (x_i, y_j) for the i^{th} step in the x direction and the j^{th} step in the y.

Now we can use the Taylor series to rewrite our derivatives.

$$\begin{aligned}\frac{\partial^2}{\partial x^2}u(x_i,y_j) &= \frac{u(x_{i+1},y_j)-2u(x_i,y_j)+u(x_{i-1},y_j)}{1/(n+1)^2}-\mathcal{O}\left(\frac{1}{(n+1)^2}\right)\\ \frac{\partial^2}{\partial y^2}u(x_i,y_j) &= \frac{u(x_i,y_{j+1})-2u(x_i,y_j)+u(x_i,y_{j-1})}{1/(n+1)^2}-\mathcal{O}\left(\frac{1}{(n+1)^2}\right).\end{aligned}$$

To obtain Poisson's equation we add the partial derivatives together to get

$$(n+1)^2\left(4u(x_i,y_j)-u(x_{i+1},y_j)-u(x_{i-1},y_j)-u(x_i,y_{j+1})-u(x_i,y_{j-1})\right)=f(x_i,y_j). \tag{1.4}$$

This is the center difference formula for the finite difference method. From here we can use the n^2 interior nodes along with the knowledge of the boundary nodes to construct a system with n^2 equations and n^2 unknowns. We then obtain a matrix equation of the form $A_{n^2}\vec{x}=\vec{b}$, with A_{n^2} a n^2 square matrix, known as the finite difference matrix. We shall work through a small example.

Example 1.2.0.1

Construct a finite difference approximation for the solution of the system

$$\begin{cases} -\Delta u(x,y) = 2\pi\sin\pi x\sin\pi y,\ 0\le x\le 1,\ 0\le y\le 1, \\ u(x,y) = 0,\ (x,0),\ (x,1),\ 0\le x\le 1,\ (0,y),\ (1,y),\ 0\le y\le 1, \end{cases} \tag{1.5}$$

with a step size of $\frac{1}{4}$ in both the x and y directions.

To solve we must construct our mesh consisting of $3^2=9$ total interior nodes on our unit square and construct our system of 9 equations and 9 unknown variables by using the Center Difference formula (1.4). Our result is

$$\begin{aligned}
(\tfrac{1}{4}, \tfrac{1}{4})\text{:} && 4u(\tfrac{1}{4}, \tfrac{1}{4}) - u(\tfrac{1}{4}, \tfrac{1}{2}) - u(\tfrac{1}{2}, \tfrac{1}{4}) &= 2\pi^2 \sin(\pi\tfrac{1}{4}) \sin(\pi\tfrac{1}{4}) \\
(\tfrac{1}{4}, \tfrac{1}{2})\text{:} && 4u(\tfrac{1}{4}, \tfrac{1}{2}) - u(\tfrac{1}{4}, \tfrac{3}{4}) - u(\tfrac{1}{4}, \tfrac{1}{4}) - u(\tfrac{1}{2}, \tfrac{1}{4}) &= 2\pi^2 \sin(\pi\tfrac{1}{4}) \sin(\pi\tfrac{1}{2}) \\
(\tfrac{1}{4}, \tfrac{3}{4})\text{:} && 4u(\tfrac{1}{4}, \tfrac{3}{4}) - u(\tfrac{1}{2}, \tfrac{1}{2}) - u(\tfrac{1}{2}, \tfrac{3}{4}) &= 2\pi^2 \sin(\pi\tfrac{1}{4}) \sin(\pi\tfrac{3}{4}) \\
(\tfrac{1}{2}, \tfrac{1}{4})\text{:} && 4u(\tfrac{1}{2}, \tfrac{1}{4}) - u(\tfrac{1}{2}, \tfrac{1}{2}) - u(\tfrac{3}{4}, \tfrac{1}{4}) - u(\tfrac{1}{4}, \tfrac{1}{4}) &= 2\pi^2 \sin(\pi\tfrac{1}{2}) \sin(\pi\tfrac{1}{4}) \\
(\tfrac{1}{2}, \tfrac{1}{2})\text{:} && 4u(\tfrac{1}{2}, \tfrac{1}{2}) - u(\tfrac{1}{2}, \tfrac{1}{4}) - u(\tfrac{1}{2}, \tfrac{3}{4}) - u(\tfrac{3}{4}, \tfrac{1}{2}) - u(\tfrac{1}{4}, \tfrac{1}{2}) &= 2\pi^2 \sin(\pi\tfrac{1}{2}) \sin(\pi\tfrac{1}{2}) \\
(\tfrac{1}{2}, \tfrac{3}{4})\text{:} && 4u(\tfrac{1}{2}, \tfrac{3}{4}) - u(\tfrac{1}{2}, \tfrac{1}{2})u(\tfrac{3}{4}, \tfrac{3}{4}) - u(\tfrac{1}{4}, \tfrac{3}{4}) &= 2\pi^2 \sin(\pi\tfrac{1}{4}) \sin(\pi\tfrac{1}{4}) \\
(\tfrac{3}{4}, \tfrac{1}{4})\text{:} && 4u(\tfrac{3}{4}, \tfrac{1}{4}) - u(\tfrac{3}{4}, \tfrac{1}{2}) - u(\tfrac{1}{2}, \tfrac{1}{4}) &= 2\pi^2 \sin(\pi\tfrac{3}{4}) \sin(\pi\tfrac{1}{4}) \\
(\tfrac{3}{4}, \tfrac{1}{2})\text{:} && 4u(\tfrac{3}{4}, \tfrac{1}{2}) - u(\tfrac{3}{4}, \tfrac{1}{4}) - u(\tfrac{3}{4}, \tfrac{3}{4}) - u(\tfrac{1}{2}, \tfrac{1}{2}) &= 2\pi^2 \sin(\pi\tfrac{3}{4}) \sin(\pi\tfrac{1}{2}) \\
(\tfrac{3}{4}, \tfrac{3}{4})\text{:} && 4u(\tfrac{1}{4}, \tfrac{1}{4}) - u(\tfrac{3}{4}, \tfrac{1}{2}) - u(\tfrac{1}{2}, \tfrac{3}{4}) &= 2\pi^2 \sin(\pi\tfrac{3}{4}) \sin(\pi\tfrac{3}{4})
\end{aligned}$$

Now we can construct a matrix equation remembering our factor of $(n+1)^2$ from the center difference formula.

$$(4)^2 \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} u(\frac{1}{4}, \frac{1}{4}) \\ u(\frac{1}{4}, \frac{1}{2}) \\ u(\frac{1}{4}, \frac{3}{4}) \\ u(\frac{1}{2}, \frac{1}{4}) \\ u(\frac{1}{2}, \frac{1}{2}) \\ u(\frac{1}{2}, \frac{3}{4}) \\ u(\frac{3}{4}, \frac{1}{4}) \\ u(\frac{3}{4}, \frac{1}{2}) \\ u(\frac{3}{4}, \frac{3}{4}) \end{bmatrix} = \begin{bmatrix} 9.8696 \\ 13.9577 \\ 9.8696 \\ 13.9577 \\ 19.7392 \\ 13.9577 \\ 9.8696 \\ 13.9577 \\ 9.8696 \end{bmatrix} \tag{1.6}$$

We then can either invert the finite difference matrix or row reduce the augmented matrix $[A_9|b]$ to obtain an estimate for the 9 interior nodes.

$$\begin{bmatrix} u(\frac{1}{4},\frac{1}{4}) \\ u(\frac{1}{4},\frac{1}{2}) \\ u(\frac{1}{4},\frac{3}{4}) \\ u(\frac{1}{2},\frac{1}{4}) \\ u(\frac{1}{2},\frac{1}{2}) \\ u(\frac{1}{2},\frac{3}{4}) \\ u(\frac{3}{4},\frac{1}{4}) \\ u(\frac{3}{4},\frac{1}{2}) \\ u(\frac{3}{4},\frac{3}{4}) \end{bmatrix} = \begin{bmatrix} 0.5265 \\ 0.7446 \\ 0.5265 \\ 0.7446 \\ 1.0530 \\ 0.7446 \\ 0.5265 \\ 0.7446 \\ 0.5265 \end{bmatrix}$$

Burden and Faires, [4], gives a local truncation error of order $\mathcal{O}(\frac{1}{(n+1)^2})$. We readily know the solution for equation (1.5) is $u(x,y) = \sin(\pi x)\sin(\pi y)$. Now we make the assumption that the error $\epsilon \sim \mathcal{O}(\frac{1}{(n+1)^s})$ and thus

$$\begin{aligned} \text{error of } n = 3 &= C\left(\frac{1}{4}\right)^s \\ \text{error of } n = 7 &= C\left(\frac{1}{8}\right)^s \\ \frac{\text{error of } n = 7}{\text{error of } n = 3} &= \left(\frac{1}{2}\right)^s \end{aligned}$$

We can compare numerical estimates with our known solution with step sizes $\frac{1}{4}$ and $\frac{1}{8}$.

Function at 9 of the n^2 interior nodes	$n=3$	$n=7$	Actual value	Error of $n=7$/ Error of $n=3$
$u(\frac{1}{4},\frac{1}{4})$	0.5265	0.5064	0.5000	0.2415
$u(\frac{1}{4},\frac{1}{2})$	0.7446	0.7162	0.7071	0.2427
$u(\frac{1}{4},\frac{3}{4})$	0.5265	0.5064	0.5000	0.2415
$u(\frac{1}{2},\frac{1}{4})$	0.7446	0.7162	0.7071	0.2427
$u(\frac{1}{2},\frac{1}{2})$	1.0530	1.0129	1.0000	0.2434
$u(\frac{1}{2},\frac{3}{4})$	0.7446	0.7162	0.7071	0.2427
$u(\frac{3}{4},\frac{1}{4})$	0.5265	0.5064	0.5000	0.2415
$u(\frac{3}{4},\frac{1}{2})$	0.7446	0.7162	0.7071	0.2427
$u(\frac{3}{4},\frac{3}{4})$	0.5265	0.5064	0.5000	0.2415

(1.7)

Note that the ratio of pointwise errors when we halve the step size is close to $\frac{1}{4}$. As we take n to be larger each time we halve the step size we see this fraction approaching $\frac{1}{4}$. This is expected since our error is $\mathcal{O}\left(\frac{1}{(n+1)^2}\right)$.

Note that our finite difference matrix in (1.6) has a nice block matrix structure. The finite difference matrix of size $(n^2 \times n^2)$, A_{n^2} is,

$$A_{n^2} = \begin{bmatrix} J & -I & & \\ -I & J & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & J \end{bmatrix} \qquad J = \begin{bmatrix} 4 & -1 & & \\ -1 & 4 & \ddots & \\ & \ddots & \ddots & \\ & & -1 & 4 \end{bmatrix}$$

for J, $(n \times n)$ and I is the $(n \times n)$ identity matrix.

1.3 Showing the finite difference method accurately approximates Poisson's equation

We know that for Poisson's equation, the operator Δ has eigenvalues and corresponding eigenfunctions,

$$\lambda_{n,m} = -(m^2+n^2)\pi^2,\ \phi_{m,n}(x,y) = \sin(m\pi x)\sin(n\pi y).$$

We want to show that as we take our step size $\frac{1}{(n+1)} \to 0$, or equivalently $n \to \infty$, that the eigenvalues of A_{n^2} approach $-(m^2+n^2)\pi^2$, the eigenvalues of Δ. From the text of Ortega [5], we have eigenvalues $\mu_{i,j}$, and eigenfunctions $x_{i,j}$ for the finite difference matrix of size $(n^2 \times n^2)$, of the form

$$\mu_{i,j} = (n+1)^2\left(4-2\left(\cos\left(\frac{i\pi}{n+1}\right)+\cos\left(\frac{j\pi}{n+1}\right)\right)\right), \tag{1.8}$$

$$x_{i,j} = \left(\sin\left(\frac{i\pi}{n+1}\right)v_j^T, \sin\left(\frac{2i\pi}{n+1}\right)v_j^T, ..., \sin\left(\frac{ni\pi}{n+1}\right)v_j^T\right), \tag{1.9}$$

where

$$v_j = \left(\sin\left(\frac{j\pi}{n+1}\right), \sin\left(\frac{2j\pi}{n+1}\right), ..., \sin\left(\frac{nj\pi}{n+1}\right)\right),\ i,j = 1,2,...,n.$$

Now we see that $\sigma(A_{n^2}) \to \sigma(\Delta)$ as $n \to \infty$.

$$\begin{aligned}
&\lim_{n\to\infty} 2(n+1)^2\left(2-\cos\left(\frac{i\pi}{n+1}\right)-\cos\left(\frac{j\pi}{n+1}\right)\right) \\
&= 2\lim_{n\to\infty}\left[\frac{1-\cos(i\pi/(n+1))}{1/(n+1)^2}+\frac{1-\cos(j\pi/(n+1))}{1/(n+1)^2}\right] \\
&= 2\lim_{n\to\infty}\left[i^2\pi^2\left(\frac{1-\cos(i\pi/(n+1))}{i^2\pi^2/(n+1)^2}\right)+j^2\pi^2\left(\frac{1-\cos(j\pi/(n+1))}{j^2\pi^2/(n+1)^2}\right)\right] \\
&= 2(i^2+j^2)\pi^2
\end{aligned}$$

It is also easy to see that the eigenvectors of the finite difference matrix, $x_{i,j}$ converge to the eigenfunctions of the Laplacian by continuity as $n \to \infty$. Therefore, a valid approximation of (1.3) is $\Delta \approx A_{n^2}$. Next consider the biharmonic system in 2-D, where the biharmonic is $\Delta^2 u = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + \frac{\partial^4 u}{\partial y^4}$,

$$\begin{cases} \Delta^2 u &= f, \text{ on } \Omega, \\ \Delta u &= 0, \text{ on } \partial\Omega, \\ u &= 0, \text{ on } \partial\Omega, \end{cases}$$

Because of the hinged boundary conditions, we can recast the biharmonic as the coupled equation

$$\begin{cases} \Delta u &= p, \text{ on } \Omega \\ u &= 0, \text{ on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} \Delta p &= f, \text{ on } \Omega \\ p &= 0, \text{ on } \partial\Omega. \end{cases}$$

By coupling the two equations we have $\Delta(\Delta w) = 0$. If we use our approximations, that is $\Delta \approx A_{n^2}$, we have $\Delta(\Delta w) \approx A_{n^2}(A_{n^2} w) = A_{n^2}^2 w$. Similarly we set out to show $\Delta^2 \approx A_{n^2}^2$. By using the center difference formula again for the biharmonic we have

$$\begin{aligned} \Delta^2 u(x,y) \approx\ & (n+1)^4[20u(x_i,y_j) - 8(u(x_{i+1},y_j) + u(x_{i-1},y_j) + u(x_i,y_{j+1}) + u(x_i,y_{j-1})) \\ & +2(u(x_{i+1},y_{j+1}) + u(x_{i+1},y_{j-1}) + u(x_{i-1},y_{j+1}) + u(x_{i-1},y_{j-1})) \\ & +(u(x_{i+2},y_j) + u(x_{i-2},y_j) + u(x_i,y_{j+2}) + u(x,y_{j-2}))]. \end{aligned}$$

Now we have that the eigenvalues of the finite-difference approximation to Δ^2 are

$$\begin{aligned} \eta &= 4(n+1)^4(2 - \cos\left(\frac{i\pi}{n+1}\right) - \cos\left(\frac{j\pi}{n+1}\right)) \\ &= \mu_{i,j}^2. \end{aligned}$$

With our coupled equation, we have that the eigenvalues of the biharmonic are the square of the eigenvalues of the Laplacian. Therefore we have $\Delta^2 \approx A_{n^2}^2$.

1.4 Constructing the control system

Consider the system

$$\begin{cases} \dot{y} &= Ay + Bu, \\ y(0) &= y_0 \end{cases}$$

Then, from the variation of parameters formula for a differential equation of this form, we have the solution for y,

$$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)}Bu(s)ds.$$

From Kalman theory, we have a useful theorem of the controllability of systems.

Theorem 1.4.1. *The system of the form*

$$\begin{cases} \dot{y} &= Ay + Bu, \\ y(0) &= y_0, \end{cases}$$

A is $(n \times n)$ and B is $(n \times m)$, is controllable if the Kalman matrix,

$$K = [B,\ AB,\ A^2B,\ ...,\ A^nB],$$

has full rank.

We wish to recast equation (1.2) in the form of $\dot{y} = Ay + Bu$. It can easily be shown that we can write the system as

$$\begin{cases} \dot{x} &= \mathcal{A}x + \mathcal{B}u, \\ x(0) &= x_0. \end{cases} \tag{1.10}$$

Where

$$x = \begin{bmatrix} w \\ w_t \end{bmatrix} \in \mathbb{R}^{2n^2}, \qquad x_0 = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in \mathbb{R}^{2n^2}, \qquad \mathcal{A} = \begin{bmatrix} 0 & I \\ -A_{n^2}^2 & -\rho A_{n^2} \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Here, $u \in L_2(0,T)$ is our control function, $\mathcal{A}$ is $(2n^2 \times 2n^2)$, and $\mathcal{B}$ is $(2n^2 \times n^2)$. Now we have our model of a vibrating plate as a control system with arbitrary initial conditions. We can check that the system (1.10) is controllable by looking at the Kalman matrix,

$$K = [\mathcal{B},\ \mathcal{AB},\ ...,\ \mathcal{A}^{2n^2}\mathcal{B}],$$

and confirming that it has full rank. This condition may be rather tedious for large n, so we shall look at K_{n^2}, the $(2n^2 \times 2n^2)$ square matrix, defined as

$$K_{n^2} = [\mathcal{B},\ \mathcal{AB}] = \begin{bmatrix} 0 & I \\ I & -\rho A_{n^2} \end{bmatrix}.$$

From the properties of the identity matrix and the finite difference matrix and the fact that the determinant for K_{n^2} is -1, K_{n^2} clearly has rank $= 2n^2$. Therefore $\text{rank}(K) = 2n^2$. Since K_{n^2} has full rank, we refer to the Kalman index, the smallest k such that $[B, AB, ..., A^k B]$ has full rank, of our system as $k = 1$. Therefore the system (1.10) is controllable.

Now we define the energy of our control u.

Definition 3. *The energy of a control u, denoted $\mathbb{E}_2(T; x_0)$ is defined by,*

$$\mathbb{E}_2(T; y_0) = \left\{ \int_0^T \|u(t; y_0)\|_2^2 dt \right\}^{1/2} .$$

with initial condition y_0.

Therefore, for R. Triggianni's suboptimal control, we can refine the control to fit our system. In this case we have $k = 1$ and therefore we define $\nu(t)$ a $2n^2$ vector by:

$$\nu(t) = \begin{bmatrix} \nu_0 \\ \nu_1 \end{bmatrix} = -K_{n^2}^{-1} e^{\mathcal{A}_{n^2} t} x_0 f_T(t),$$

where $f_T(0) = f_T(T) = 0$ and $\int_0^T f_T(t)dt = 1$. Thus the $(n^2 \times 1)$ suboptimal control is

$$u_{x_0}(t) = \nu_0 + \frac{d}{dt}(\nu_1),$$

And its energy

$$\begin{aligned} \mathbb{E}_2(T; x_0) &= \left\{ \int_0^T \|u_{x_0}(t)\|_2^2 dt \right\}^{\frac{1}{2}} \\ &\geq \mathbb{E}_2^0(T) = \sup_{\|y_0\|=1} \mathbb{E}_2^0(T; y_0). \end{aligned}$$

We wish to estimate the behavior of the minimal energy as we take $T \to 0$. So we set out to prove the following theorem:

Theorem 1.4.2. *For the system (1.10) the energy of the control, denoted $\mathbb{E}_2^x$, of $u_{x_0}(t)$ defined above, steering the initial state $[w_0, w_1]$ to 0 at time $t = T$, obeys the* *estimate*

$$
\begin{aligned}
\mathbb{E}_2^x(T;x_0) &= \left\{\int_0^T \|u_{x_0}(t;x_0)\|_2^2 dt\right\}^{\frac{1}{2}} \\
&= \mathcal{O}\left(T^{-\frac{3}{2}}\right), \quad as\ T \to 0.
\end{aligned}
$$

And thus,

$$
\mathbb{E}_2^0(T;x_0) = \mathcal{O}\left(T^{-\frac{3}{2}}\right), \quad as\ T \to 0.
$$

Chapter 2

Bounding The Control

We set out to bound Triggianni's sub-optimal control $\|u_{x_0}\|_2$ in space, such that

$$\|u_{x_0}\|_2 \leq CT^{-\frac{3}{2}},$$

where C is a constant independent of n. To do so, we shall find suitable bounds for $\|K_{n^2}^{-1}\|_2$, $\|e^{\mathcal{A}t}\|_2$, and $\|\mathcal{A}e^{\mathcal{A}t}\|_2$. It is important to find our bounds independent of n so that we may make more accurate finite difference approximations with large n without effecting our bounds.

2.1 $\|K^{-1}\|_2$

Consider the transformation to our system (1.10), by $[y, y_t] \to [Aw, w_t]$. Equivalently,

$$\frac{d}{dt}\begin{bmatrix} w \\ w_t \end{bmatrix} = \begin{bmatrix} A_{n^2}^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} 0 & I \\ -A_{n^2}^2 & -\rho A_{n^2} \end{bmatrix}\begin{bmatrix} A_{n^2} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} w \\ w_t \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u \quad (2.1)$$

$$\frac{d}{dt}\begin{bmatrix} A_{n^2}w \\ w_t \end{bmatrix} = \begin{bmatrix} 0 & A_{n^2} \\ -A_{n^2} & -\rho A_{n^2} \end{bmatrix}\begin{bmatrix} A_{n^2}w \\ w_t \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u. \quad (2.2)$$

Notice that our system (2.1) can still be written as $\dot{x} = \mathcal{A}x + \mathcal{B}u$, and our Kalman index $k = 1$ of the Kalman matrix

$$K_{n^2} = \begin{bmatrix} 0 & A_{n^2} \\ I & -\rho A_{n^2} \end{bmatrix},$$

is still preserved. It can be easily verified that its inverse is

$$K_{n^2}^{-1} = \begin{bmatrix} \rho I & I \\ A_{n^2}^{-1} & 0 \end{bmatrix}.$$

We want to find a bound on $\|K_{n^2}^{-1}\|_2$ that is independent of n. To do so, we make use of Theorem (2.5.6) from Ortega [5] which gives

$$\|K_{n^2}^{-1}\|_2^2 = \rho\left((K_{n^2}^{-1})^T K_{n^2}^{-1}\right),$$

where $\rho(L) = \max_{1\le i\le n} |\lambda_i|$, $\lambda_i \in \sigma(L)$. With that in mind we write

$$\begin{aligned}[K_{n^2}^{-1}]^T K_{n^2}^{-1} &= \begin{bmatrix} \rho I & A_{n^2} \\ I & 0 \end{bmatrix} \begin{bmatrix} \rho I & I \\ A_{n^2}^{-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} \rho^2 + A_{n^2}^{-2} & \rho I \\ \rho I & I \end{bmatrix}. \end{aligned}$$

Next, let λ be an eigenvalue of $[K_{n^2}^{-1}]^T K_{n^2}^{-1}$, with corresponding eigenvector $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2n^2}$. By definition we have,

$$\begin{bmatrix} \rho^2 + A_{n^2}^{-2} & \rho I \\ \rho I & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix},$$

and hence,

$$\begin{cases} (\rho^2 + A_{n^2}^{-2})x + \rho y = \lambda x \\ \rho x + y = \lambda y \end{cases} \tag{2.3}$$

From the second equation we obtain that $y = \left(\frac{\rho}{\lambda - 1}\right) x$, and from the first equation, we see that $\lambda \neq 1$. Using our solution for y we solve for x in (2.3).

$$(\rho^2 + A_{n^2}^{-2})x + \left(\frac{\rho^2}{\lambda - 1}\right) x = \lambda x.$$

Now we let x be an eigenvector of A_{n^2}, that is $x = x_{i,j}$, where $x_{i,j}$ is defined as before in (1.8). So we can now write

$$
\begin{array}{rcl}
(\rho^2 + \mu_{i,j}^{-2})x + \left(\dfrac{\rho^2}{\lambda - 1}\right)x & = & \lambda x, \\
\implies \qquad (\lambda^2 - \lambda)x + (1-\lambda)(\rho^2 + \mu_{i,j}^{-2})x - \rho^2 x & = & 0, \\
\implies \qquad (\lambda^2 - \lambda(1 + \rho^2 + \mu_{i,j}^{-2}) + \mu_{i,j}^{-2})x & = & 0.
\end{array}
$$

So we have that the eigenvalues of $[K_{n^2}^{-1}]^T K_{n^2}^{-1}$ are given by the quadratic equation as

$$
\lambda_{i,j} = \frac{1 + \rho^2 + \mu_{i,j}^{-2}}{2} \pm \frac{\sqrt{(1 + \rho^2 + \mu_{i,j}^{-2})^2 - 4\mu_{i,j}^{-2}}}{2} \qquad i, j = 1, 2, ..., n.
$$

From our eigenvalues (1.8), we have our bound

$$
\|K_{n^2}^{-1}\|_2 \leq r_{0,\rho}.
$$

Where $r_{0,\rho}$ constant, can be chosen independent of n.

2.2 $\|e^{\mathcal{A}t}\|_2$

Again, we consider our transformed system (2.1), and wish to show that

$$
\|e^{\mathcal{A}t}\| \leq r_{1,\rho}.
$$

For $r_{1,\rho}$ constant and independent of n. To find the eigenpairs of $\mathcal{A} = \begin{bmatrix} 0 & A_{n^2} \\ -A_{n^2} & -\rho A_{n^2} \end{bmatrix}$ we refer to the factorization in [3] to see that the eigenpairs are

$$
\left\{ z_1 \mu_{i,j}, \begin{bmatrix} x_{i,j} \\ z_1 x_{i,j} \end{bmatrix} \right\} \cup \left\{ z_2 \mu_{i,j}, \begin{bmatrix} x_{i,j} \\ z_2 x_{i,j} \end{bmatrix} \right\}, \tag{2.4}
$$

where z_1 and z_2 are the solutions to the quadratic polynomial $\lambda^2+\rho\lambda+1$. Since we restricted $\rho > 2$ at the beginning of this paper then $z_1 = \frac{-\rho+\sqrt{\rho^2-4}}{2}$, $z_2 = \frac{-\rho-\sqrt{\rho^2-4}}{2}$, $z_1 + z_2 = -\rho$, $z_1 z_2 = 1$, $z_i < 0,\ i = 1, 2$ and are real numbers.

Note that each (n^2-vector) $x_{i,j}$ satisfies

$$\langle x_{i,j}, x_{\tilde{i},\tilde{j}} \rangle = \begin{cases} 0, & \text{if } i \neq \tilde{i} \text{ or } j \neq \tilde{j}, \\ c_n = \frac{n+1}{2}, & \text{if } i = \tilde{i} \text{ or } j = \tilde{j}. \end{cases}$$

Now denote,

$$\begin{aligned} \Phi_{i,j} &= \frac{1}{\sqrt{c_n(1+z_1^2)}} \begin{bmatrix} x_{i,j} \\ z_1 x_{i,j} \end{bmatrix} \ i, j \leq n, \\ \Psi_{i,j} &= \frac{1}{\sqrt{c_n(1+z_2^2)}} \begin{bmatrix} x_{i,j} \\ z_2 x_{i,j} \end{bmatrix} \ i, j \leq n, \end{aligned}$$

then $\{\Phi_{i,j}\}$ and $\{\Psi_{i,j}\}$ are each orthonormal and furthermore $\{\Phi_{i,j}, \Psi_{i,j}\}$ are bi-orthonormal. Hence, for any $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2n^2}$, we can find the unique expansion,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha_{i,j}\Phi_{i,j} + \beta_{i,j}\Psi_{i,j}. \tag{2.5}$$

From equation (2.5), we have

$$\begin{aligned} \langle \begin{bmatrix} x \\ y \end{bmatrix}, \Phi_{i,j} \rangle &= \alpha_{i,j} + \frac{c_n(1+z_2 z_1)}{\sqrt{c_n(1+r_2^2)}\sqrt{c_n(1+z_1^2)}}\beta_{i,j} \\ &= \alpha_{i,j} + \frac{2}{\sqrt{(1+z_2^2)(1+z_2^2)}}\beta_{i,j}. \end{aligned}$$

Likewise,

$$\langle \begin{bmatrix} x \\ y \end{bmatrix}, \Psi_{i,j} \rangle = \frac{2}{\sqrt{(1+z_2^2)(1+z_1^2)}} \alpha_{i,j} + \beta_{i,j}.$$

So for $i, j = 1, 2, ..., n$,

$$L_\rho \begin{bmatrix} \alpha_{i,j} \\ \beta_{i,j} \end{bmatrix} = \begin{bmatrix} \langle \begin{bmatrix} x \\ y \end{bmatrix}, \Phi_{i,j} \rangle \\ \langle \begin{bmatrix} x \\ y \end{bmatrix}, \Psi_{i,j} \rangle \end{bmatrix}. \tag{2.6}$$

For some operator L_ρ, dependent on ρ that has an inverse. Note that $e^{\mathcal{A}t}$ can be written as $e^{\mathcal{A}t} = I + \frac{\mathcal{A}t}{1!} + \frac{A^2t^2}{2!} + ... = \sum_{k=1}^{\infty} \frac{\mathcal{A}t}{k!}$. therefore, for $\Phi_{i,j}$ we have $e^{\mathcal{A}t}\Phi_{i,j} = \Phi_{i,j} + \frac{A\Phi_{i,j}t}{1!} + ... = \Phi_{i,j} + \frac{z_1\mu_{i,j}\Phi_{i,j}t}{1!} + ... = e^{z_1\mu_{i,j}t}\Phi_{i,j}$. Thus for $[x, y]^T \in \mathbb{R}^{2n^2}$, $[x, y]^T \neq 0$, and the fact that $(x + y)^2 \leq (x^2 + y^2)$ for all x, y,

$$\begin{aligned} \left\| e^{\mathcal{A}t} \begin{bmatrix} x \\ y \end{bmatrix} \right\|_2^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} \left\{ \alpha_{i,j}^2 e^{2z_1\mu_{i,j}t} + 2\alpha_{i,j}\beta_{i,j}e^{-\rho\mu_{i,j}t} \left[\frac{2}{\sqrt{(1+z_1^2)(1+z_2^2)}} \right] + \beta_{i,j}^2 e^{2z_2\mu_{i,j}t} \right\} \\ &\leq e \sum_{i=1}^{n}\sum_{j=1}^{n} [\alpha_{i,j}^2 + \beta_{i,j}^2]. \end{aligned} \tag{2.7}$$

From equation (2.6) we obtain for $i, j = 1, 2, .., n$,

$$\left\| \begin{bmatrix} \alpha_{i,j} \\ \beta_{i,j} \end{bmatrix} \right\|_2^2 \leq r_{1,\rho} \left\{ \langle \begin{bmatrix} x \\ y \end{bmatrix}, \Phi_{i,j} \rangle^2 + \langle \begin{bmatrix} x \\ y \end{bmatrix}, \Psi_{i,j} \rangle^2 \right\} \tag{2.8}$$

Now,

$$\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \Phi_{i,j} \right\rangle^2 = \frac{\langle x, x_{i,j}\rangle^2}{c_n(1+z_1^2)} + \frac{\langle y, z_1 x_{i,j}\rangle^2}{c_n(1+z_2^2)} \tag{2.9}$$

$$\leq r_{1,\rho}\left(\frac{\langle x, x_{i,j}\rangle^2}{c_n} + \frac{\langle y, x_{i,j}\rangle^2}{c_n}\right) \tag{2.10}$$

Likewise,

$$\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \Psi_{i,j} \right\rangle^2 \leq r_{1,\rho}\left(\frac{\langle x, x_{i,j}\rangle^2}{c_n} + \frac{\langle y, x_{i,j}\rangle^2}{c_n}\right). \tag{2.11}$$

By combining equations (2.8) through (2.11) we have,

$$\left\| \begin{bmatrix} \alpha_{i,j} \\ \beta_{i,j} \end{bmatrix} \right\|_2^2 \leq r_{1,\rho}\left(\frac{\langle x, x_{i,j}\rangle^2}{c_n} + \frac{\langle y, x_{i,j}\rangle^2}{c_n}\right). \tag{2.12}$$

Applying (2.12) to the right hand side (RHS) of (2.7), we obtain for $t > 0$,

$$\left\| e^{\mathcal{A}t} \begin{bmatrix} x \\ y \end{bmatrix} \right\|_2^2 \leq r_{1,\rho} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ \frac{\langle x, x_{i,j}\rangle^2}{c_n} + \frac{\langle y, x_{i,j}\rangle^2}{c_n} \right\}.$$

Since $\left\{\frac{1}{\sqrt{c_n}} x_{i,j}\right\}$ is an orthonormal basis of $\mathbb{R}^{n^2}$, we have by Parseval's Relation that for all $[x, y]^T \in \mathbb{R}^{2n^2}$,

$$\left\| e^{\mathcal{A}t} \begin{bmatrix} x \\ y \end{bmatrix} \right\|_2^2 \leq r_{1,\rho} \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|_2^2.$$

By the definition of the matrix norm, we have,

$$\|e^{\mathcal{A}t}\|_2 \leq r_{1,\rho}, \text{ for all } t > 0.$$

2.3 $\|\mathcal{A}e^{\mathcal{A}t}\|_2$

We shall proceed as before with the transformation in system (2.1). Again, the eigenpairs are given by (2.4), and we have $\Phi_{i,j}$ and $\Psi_{i,j}$ which are each orthonormal, and together are biorthonormal. Thus,

$$\begin{aligned} \left\| \mathcal{A}e^{\mathcal{A}t} \begin{bmatrix} x \\ y \end{bmatrix} \right\|_2^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} \{ z_1^2 \mu_{i,j}^2 \alpha_{i,j}^2 e^{2z_1\mu_{i,j}t} \\ &+ 2\mu_{i,j}^2 \alpha_{i,j} \beta_{i,j} e^{-\rho\mu_{i,j}t} \left(\frac{2}{\sqrt{(1+z_1^2)(1+z_2^2)}} \right) + z_2^2 \mu_{i,j}^2 \beta_{i,j}^2 e^{2z_2\mu_{i,j}t} \} \end{aligned}$$

We wish to bound these sums by using the following.

$$\begin{aligned} z_1^2 \mu_{i,j}^2 + z_2^2 \mu_{i,j}^2 &\leq z_1^2 \mu_{i,j}^2 + 2\mu_{i,j}^2 + z_2^2 \mu_{i,j}^2 = \mu_{i,j}^2 (z_1 + z_2)^2 = \rho^2 \mu_{i,j}^2 \\ e^{2z_k\mu_{i,j}t} &\leq e^{-2|z_1|\mu_{i,j}} \text{ for all } k = 1, 2 \text{ and } i, j = 1, \ldots, n, \ t \in (0, T]. \end{aligned}$$

So therefore,

$$\left\| \mathcal{A}e^{\mathcal{A}t} \begin{bmatrix} x \\ y \end{bmatrix} \right\|_2^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \rho^2 \mu_{i,j}^2 e^{-2|z_1|\mu_{i,j}t} (\alpha_{i,j}^2 + \beta_{i,j}^2).$$

To obtain a bound on the series above, consider the function

$$f(x) = \rho^2 x^2 e^{-2|z_1|xt}, \ x > 0,$$

for $f(x)$ a function of x, the eigenvalues of the finite difference matrix. This function is an exponentially decaying function. By using calculus and taking its derivative we can find a maximum to the function.

$$\frac{df}{dx} = (2\rho^2 x - 2|z_1|t\rho^2 x^2)e^{-2|z_1|xt} = 2\rho^2 x(1 - |z_1|tx)e^{-2|z_1|xt} = 0.$$

Therefore our critical points are $x = 0$ and $x = \frac{1}{|z_1|t}$. We clearly have a minimum at $x = 0$ and a maximum is attained $f(\frac{1}{|z_1|t})$. Also note that by the definition, z_i has dependence on ρ and is independent of n. So we have,

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} \rho^2 \mu_{i,j}^2 e^{-2|z_1|\mu_{i,j}t}(\alpha_{i,j}^2 + \beta_{i,j}^2). &\leq \sum_{i=1}^{n}\sum_{j=1}^{n} \rho^2 \left(\frac{1}{|z_1|t}\right)^2 e^{-2|z_1|t(\frac{1}{|z_1|t})}(\alpha_{i,j}^2 + \beta_{i,j}^2) \\ &= \frac{\rho^2}{|z_1|^2 t^2 e^2} \sum_{i=1}^{n}\sum_{j=1}^{n} (\alpha_{i,j}^2 + \beta_{i,j}^2)\end{aligned}$$

Again we refer to the bound on $\|[\alpha_{i,j}\ \beta_{i,j}]^T\|_2$ that we obtained from (2.8), and we use Parseval's Relation as before to obtain

$$\begin{aligned}\left\|\mathcal{A}e^{\mathcal{A}t}\begin{bmatrix} x \\ y \end{bmatrix}\right\|_2 &\leq \frac{\rho^2}{|z_1|^2 t^2 e^2}\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\|_2^2 \\ &\leq \frac{r_{2,\rho}}{t^2}\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\|_2^2.\end{aligned}$$

Which gives us the bound

$$\left\|\mathcal{A}e^{\mathcal{A}t}\begin{bmatrix} x \\ y \end{bmatrix}\right\|_2 \leq \frac{r_{2,\rho}}{t}\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\|_2$$

for some constant $r_{2,\rho}$ independent of n.

2.4 Bounding $\|u^x(t;x_0)\|_2$

With reference to our suboptimal control $u_{x_0}(t) = \nu_0(t) + \nu_1'(t)$, where

$$\nu(t) = \begin{bmatrix} \nu_0(t) \\ \nu_1(t) \end{bmatrix} = -K_{n^2}^{-1} e^{\mathcal{D}t} x_0 f_T(t)$$

we let $f_T(t) = (6/T^3)t(T-t)$, which satisfies the criteria $f_T(0) = f_T(T) = 0$, and is normalized such that $\int_0^T f_T(t)dt = 1$.

Notice the control is made up of the terms $[-K^{-1}e^{\mathcal{A}t}x_0 f_T(t)]$, and $[-K^{-1}\mathcal{A}e^{\mathcal{A}t}x_0 f_T(t) - K^{-1}e^{\mathcal{A}t}x_0 \left(\frac{d}{dt}f_T(t)\right)]$. We wish to bound the energy of our control, first by using

$$\begin{aligned} \mathbb{E}_2^x(x_0;T) &= \left\{\int_0^T \|u(t)\|_2 dt\right\}^{1/2} \\ &\leq \left\{\int_0^T dt\right\}^{1/2} \|u(t)\|_2,\ for\ 0 < t \leq T. \end{aligned} \tag{2.13}$$

From our control the term that gives the worst estimate is $\|-K^{-1}e^{\mathcal{A}t}\left(\frac{d}{dt}f_T(t)\right)\|_2$ for $0 < t \leq T$. By using our bounds and our definition of $f_T(t)$ we have

$$\begin{aligned} \|u_{x_0}(t)\|_2 &\leq r_{0,\rho}r_{1,\rho}\left|\frac{6(T-2t)}{T^3}\right| \|x_0\|_2,\ 0 < t \leq T, \\ &\leq C_{\rho,r}\frac{6}{T^2}\|x_0\| \leq C_{\rho,r}T^{-2}\|x_0\|_2. \end{aligned}$$

Now by using inequality (2.13), we have

$$\begin{aligned} \mathbb{E}_2(x_0;T) &\leq \left\{\int_0^T dt\right\}^{1/2} \|u(t)\|_2\ 0 < t \leq T, \\ &\leq D_{\rho,\gamma}\frac{T^{1/2}}{T^2}\|x_0\| = D_{\rho,\gamma}T^{-3/2}\|x_0\|_2. \end{aligned}$$

Where $D_{\rho,\gamma}$ is independent of n. Since we have found a uniform bound independent of n, we have, by definition,

$$\mathbb{E}_2(T; x_0) = \mathcal{O}\left(T^{-\frac{3}{2}}\right).$$

Which completes the proof.

Chapter 3

Conclusion

We see that the norm of our energy goes to infinity no faster than $\mathcal{O}(T^{-\frac{3}{2}})$. To see that this estimate is sharp we rely on numerics to provide some insight. So we can numerically estimate the rate, as $T \to 0$. We shall use the assumption that $\mathbb{E}_2(T, y_0) \sim T^{-s}$, for some s. So we have

$$\begin{aligned} \mathbb{E}_2(T^k, x_0) &= CT^{-ks} \\ \frac{\mathbb{E}_2(T^k, x_0)}{\mathbb{E}_2(T^{k+1}, x_0)} &= \frac{CT^{-ks}}{CT^{-(k+1)s}} = T^{-s} \\ s &= -\frac{log(\mathbb{E}_2(T^k, x_0))}{log(\mathbb{E}_2(T^{k+1}, x_0))log(T)}. \end{aligned}$$

Terminal Time	Energy Estimate	Estimate on s for $\mathcal{O}(T^{-s})$
$\left(\frac{1}{2}\right)^0$	34.022	NA
$\left(\frac{1}{2}\right)^1$	92.046	1.4358
$\left(\frac{1}{2}\right)^2$	250.56	1.4447
$\left(\frac{1}{2}\right)^3$	691.45	1.4644
$\left(\frac{1}{2}\right)^4$	1928.7	1.4799
$\left(\frac{1}{2}\right)^5$	5315.1	1.4893
$\left(\frac{1}{2}\right)^6$	15258.0	1.4945
$\left(\frac{1}{2}\right)^7$	43074.0	1.4972
$\left(\frac{1}{2}\right)^8$	121710	1.4985
$\left(\frac{1}{2}\right)^9$	344090	1.4993

(3.1)

Above is a table with various terminal times T, with the estimate for s given by the algorithm in (3.1).

Now we discuss the choice of $\rho > 2$. When we refer to the factorization in [3], we have eigenvalues and corresponding eigenvectors of,

$$\left\{ z_1\mu_{i,j}, \begin{bmatrix} x_{i,j} \\ z_1 x_{i,j} \end{bmatrix} \right\} \cup \left\{ z_2\mu_{i,j}, \begin{bmatrix} x_{i,j} \\ z_2 x_{i,j} \end{bmatrix} \right\},$$

where z_1 and z_2 are solutions to the quadratic polynomial $\lambda^2 + \rho\lambda + 1$. For $\rho > 2$ we have z_1 and z_2 real and negative which helps us bound $\|e^{\mathcal{A}t}\|_2$ and $\|\mathcal{A}e^{\mathcal{A}t}\|_2$ because we have exponential decay with the eigenvalues. If $\rho < 2$ we have z_1 and z_2 with negative real parts of our complex z_i's. Therefore our results still hold. With $\rho = 2$, the critical case, we have a whole different situation that R. Triggianni doesn't cover in [6], and says that the same result holds.

Chapter 4

Appendix

MATLAB code for the Finite Difference Method. (4.1)

```
function [u,A,unew,error,B,U,W,E]=FDLaS(n,m,a,b,c,d)

%FDLaS is a finite differencing method to approximate
%the solution of the
%Laplace equation with a source term g. Input n
%partitions in the x
%direction %from a to b. m partitions in the y direction
%from c to d.
%Don't forget the boundary conditions zxy, zyx, maxxy, maxyx.

B=zeros((n-1)^2,1);
U=zeros(n+1,m+1);
W=zeros(n,m);
```

```
A=zeros((n-1)^2,(m-1)^2);
h=(b-a)/n;
k=(d-c)/m;
E=zeros((n-1),(m-1));

%E is a matrix used for the weird ordering that is used in the
%Burden and
%Faires book for the Finite Differencing of the Laplacian.

for i=1:(n-1)
    for j=1:(m-1)
        E(i,j)=j+(i-1)*size(E,1);
    end
end

%The next lines create the square plate and set it's boundary
%conditions.

for i=1:(n+1)
    U(i,1)=zxy(d-(i-1)*k);
    U(i,m+1)=maxxy(d-(i-1)*k);
end
for i=2:m
    U(1,i)=maxyx((i-1)*h);
    U(n+1,i)=zyx((i-1)*h);
```

```
end


%W is also used for ordering.

for i=1:(n)
    for o=1:(m)
        W(i,o)=(i)+(m-1-o)*(n-1);
    end
end

%B is the right hand of the equation Kf=g. So it contains the
%source term
%and the boundary terms for the Laplacian.

for i=2:size(U,1)-1
    for j=2:size(U,2)-1
    B(E(i-1,j-1))=((h/k)^2)*(U(i+1,j)+U(i-1,j))+(U(i,j+1)+U(i,j-1))
        -h^2*g(a+(j-1)*h,d-(i-1)*k);
    end
end

%Finally, A is the Coefficient matrix for the Finite Difference problem.

for i=1:(size(W,1)-1)
```

```
        for o=1:(size(W,2)-1)
            for l=1:(size(W,1)-2)
                for p=2:(size(W,1)-1)
                A(W(i,o),W(i,o))=2*((h/k)^2+1);
                A(W(i,l),W(i,l+1))=-(h/k)^2;
                A(W(i,p),W(i,p-1))=-(h/k)^2;
                A(W(l,i),W(l+1,i))=-1;
                A(W(p,i),W(p-1,i))=-1;
                end
            end
        end
end

%The next operations finds the approximations at the nodes,
%then reorders
%them to match up with Mike's answers.

u=A\B;
M=zeros(n-1);
for i=1:(n-1)
    M(i,:)=u((i-1)*(n-1)+1:(i*(n-1)))';
end
M=M';
M=flipud(M);
NewP=zeros(n+1);
```

```
NewP(2:n,2:n)=M;
Ptran=NewP';
unew=zeros(n+1,1);
for i=2:(n+1)
    unew=[flipud(Ptran(:,i));unew];
end

uact=sin(pi*linspace(0,1,n+1)')*sin(pi*linspace(0,1,n+1));

ucomp=zeros(n+1,1);
for i=2:n+1
    ucomp=[ucomp;uact(i,:)'];
end
[unew, ucomp];
error = (unew-ucomp);

subplot(2,1,2);
surf(linspace(0,1,n+1)',linspace(0,1,n+1)',abs(uact-Ptran));
subplot(2,1,1);
surf(linspace(0,1,n+1)',linspace(0,1,n+1)',Ptran);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function g=g(x,y)
```

```
g=-2*pi^2*sin(pi*x)*sin(pi*y);
%g=-64*(-72*(1-5*x+5*x^2)*sin(pi*y)-12*pi^2*x*(1-x)*(1-5*x+5*x^2)
%*sin(pi*y)+
% pi^4*x^3*(1-x)^3*sin(pi*y));
%g=-8^4*(-72*(1-5*x+5*x^2)*y^3*(1-y)^3-72*(1-5*y+5*y^2)*x^3*(1-x)^3+
% 72*x*(1-x)*(1-5*x+5*x^2)*y*(1-y)*(1-5*y+5*y^2));


% Test for FDLaS...To see if we're correct.
% It works for this example.


%g = (x^2 + y^2)*exp(x*y);


end


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


function maxyx=maxyx(x)
maxyx=0;
end


function maxxy=maxxy(y)
maxxy=0;
end


function zxy=zxy(y)
```

```
zxy=0;
end

function zyx=zyx(x)
zyx=0;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function retval = rate_error(m)

% Computes the error rates for finite difference
% method using FDLaS.m, g.m, and the boundaries.

% Call the program to create the finite difference
% matrix A, it's solution u, and pointwise error
% 'error'.
 [u,A,unew,error] = FDLaS(4,4,0,1,0,1);

% Set the maximum pointwise error to error1.
  error1 = max(error);

% Use algorithm along with assumption that
% error = O(1/n^s), and compute errors using
% powers of 2 for n.
```

```
for i = 1:m

    [u,A,unew,error2] = FDLaS(2^(2+i),2^(2+i),0,1,0,1);
    error2 = max(error2);
    disp(sprintf('error %d = %f',i,(-log(error2/error1))*(1/log(2.0))))

    error1 = error2;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

MATLAB code to solve system (1.10). (4.2)

```
%To solve y'(t)=Ay(t)+Bu(t), we get y(t+1)=(dtA+I)y(t)+dtBu(t)

function y=AyBuDeltaE(A1,B1,y0,T00)

% Function to solve y' = Ay+Bu using the variation of parameters
% formula. Used to check if the control works by computing
% y(T) and checking terminal condition.

y=y0; T=100; dt=T00/T; A=A1; B=B1;coeff=((factorial(size(A,1)...
    -1))^2/(factorial(2*(size(A,1))-1)))*T00^(2*(size(A,1))-1);
tic
for t=1:T
```

```
    y=vpa(dt*A*y+y+dt*B*ufun(A,B,y0,T00,coeff,t*dt),5);
end
toc
end
```

MATLAB code to compute rate of blowup. (4.3)

```
function [retval1,retval2]=rate_blowup(n)

%This function finds the control u by using Trigianni's method
%in finding
%an exact null controler from paper. We then integrate it and
%find s by
%setting s equal to ln(E1/E2)/ln(2).

%First we must construct A which is defined in Trigianni's paper
%as
%[0          I ]
%[-K   -pK^(1/2)]
%Where K is the Finite Difference matrix for the Laplacian using
%FDLaS and
%K^2 is the Finite Difference matrix for the Biharmonic using
%vortex or
%just squaring the FDLaS matrix, depending on what example you want.
```

```
%Construct y0 which is the vector of ones.

y0=ones(2*(n-1)^2,1);

%Construction of K=k

[u,a]=FDLaS(n,n,0,1,0,1);
k=a

%Construction of K^2=K

    %Using FDLaS twice

    K=k^2;

    %All at once

    %K=k^2+diag(20*ones(size(k^2,1),1)-diag(k^2));

%Now construct the 0=z and I=I matrices.

z=zeros((n-1)^2,(n-1)^2);
I=eye((n-1)^2);

%And finally big A.
```

```
A=[z  I;-K -k];
size(A)

%Now construct B which by definition is
%[0]
%[I]

B=[z;I];
retval1=[B A*B];
retval2=[k  I;  I  z];



%retval=[B A*B];

%Now we must integrate u to find the Energy

Energy=zeros(10,1);
for  i=1:10
     T=(1/2)^i;
     %coeff=((factorial(size(A,1)-1))^2/(factorial(2*(size(A,1))-1)))
     %*T^(2*(size(A,1))-1);
     coeff = (6/(T^3));
     %Now integrate by using the trapazoidal rule.
```

```
    tic

    trap=0;
    for j=1:9
        trap=2*ufun(A,B,y0,T,coeff,(j*(T)/(10))).^2+trap;
    end
    trapfin=(trap+ufun(A,B,y0,T,coeff,0).^2+ufun(A,B,y0,T,coeff,T).^2)..
*T/(2*10);

    toc

    %Now we take the square root and put it into a vector.

    Energy(i)=double(vpa(sqrt(sum(trapfin)),5));

end

%Now we can calculate s by using natural logs and what not.

s=zeros(9,1);

for k=1:9
    s(k)=-log(Energy(k)/Energy(k+1))*(1/log(2.0));
end
```

```
Energy
retval1=s;



%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function [K_li,K,k]=kalman(A,B)

% Constructs the Kalman matrix with least Kalman index

k=0;
K=B;
K_li=zeros(size(A));
if rank(K)<size(A,1) & size(A,2)==size(B,1)
    for i=1:size(A,1)-1
        K=[K A^i*B];
        if rank(K)==size(A,1) & k==0
            k=i;
            K_li=B(:,1);
            for j=2:size(K,2)
                if rank([K_li K(:,j)])>rank(K_li)
                    K_li=[K_li K(:,j)];
                end
            end
        end
```

```
    end
end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function retval=trap(A,B,a,b,n,m,coeff)

%uses the trapazoidal rule to integrate a function over
%a to b with n subdivisions.

x=linspace(a,b,m);

%input function here to integrate

fun=ufunlist(A,B,ones(n-1),ones((n-1)^2)*b,coeff,x);
fun2=fun*2;
fun2(:,1)=fun2(:,1)/2;
fun2(:,n)=fun2(:,n)/2;
retval=(sum(fun2'))'*(b-a)/(2*n);

%Now we compute error.

syms y;
fun3=[y; 4*y^4; y^3+exp(y)];
```

```
actual=vpa(int(fun3,y,a,b),5)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function u=u(A,B,y0,T,coeff,t)

% Computes the sub-optimal control given by Triggianni in "paper"

% call kalman which constructs the Kalman matrix with least
% Kalman index, in our case k=1.
k=kalman1(A,B);

u1=[eye(size(B,2),size(B,2)) zeros(size(B,2),size(B,2))]*vpa((inv(k)...
    *(-expm(A*t)*y0*(t*(T-t)/coeff))),5);
u2=[zeros(size(B,2),size(B,2)) eye(size(B,2),size(B,2))]*vpa(inv(k)...
    *(-expm(A*t)*y0*coeff*(T-2*t)-A*expm(A*t)*y0*(coeff*t*(T-t))));
u=vpa(u1+u2,5);
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

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