# Modeling physical system with randomness using stochastic differential equations

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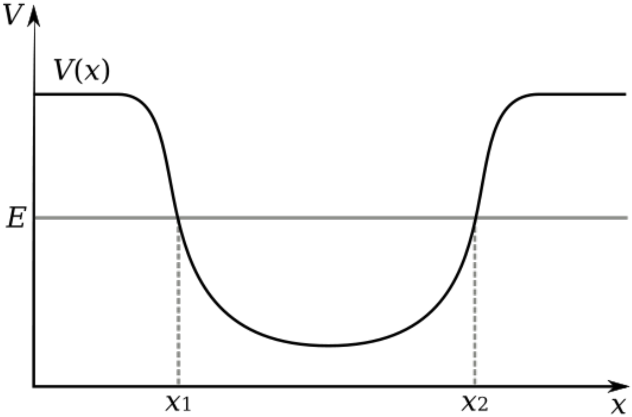
# Modeling with SDEs




(en.wikipedia.org)

# SDE applications

- Physics (lasers, scattering of slow neutrons, magnetic resonance, light scattering, ect.)
- Biology (gene mutations, population dynamics)
- Finance
- Economics
- Structural Engineering (seismic activity)

# ODEs and SDEs

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$$m\frac{d^2x(t)}{dt^2} = b(x(t), \dot{x}(t), t) + \sigma(x(t), t)\eta(t),$$

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- First take $\eta(t)$ to have equal chance, pos/neg with magnitude 1, and independent for each $t$. (White Noise).

- As a system,

$$\begin{aligned} dx(t) &= v(t)\ dt \\ dv(t) &= b(x(t), v(t), t)\ dt + \sigma(x(t), t)\ dW_t, \end{aligned}$$

where $W_t$ is a standard Wiener process (Brownian motion).

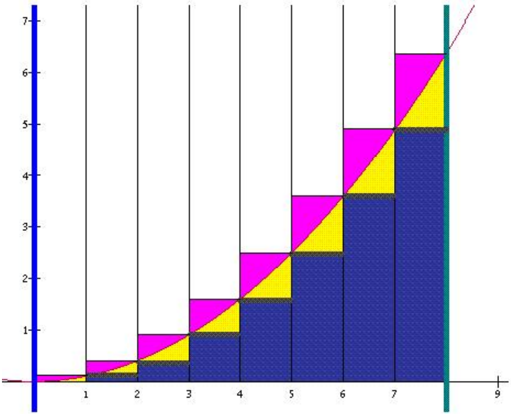
$$d\mathbf{x}(t) = B(\mathbf{x}(t), t)\ dt + \Sigma(\mathbf{x}(t), t)\ d\mathbf{W}_t.$$

Which is shorthand for

$$\mathbf{x}(t) = \int_0^t B(\mathbf{x}(s), s)\ ds + \int_0^t \Sigma(\mathbf{x}(s), s)\ d\mathbf{W}_s.$$

- How do we define the second integral?

# Riemann Sums




(conservapedia.com)

- Helpful to think in terms of discrete increments. For

$$dx_t = b(x_t)\, dt$$

- Use Euler solver to solve for $x(t)$.

$$x(t_{n+1}) = x(t_n) + b(x(t_n))(t_{n+1} - t_n)$$

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$$dx(t) = b(x(t))\ dt + \sigma\ dW_t$$

$b(x(t))$ is called the **drift** and $\sigma$ is **diffusion** coefficients.

- First generate a Wiener process by

$$W(t_{n+1}) = W(t_n) + \sqrt{(t_n - t_{n-1})}\text{randn} \quad (\text{Gaussian RV} \in [-1, 1])$$

- Then use Euler solver to solve for $x(t)$.

$$x(t_{n+1}) = x(t_n) + b(x(t_n))(t_n - t_{n-1}) + \sigma(W(t_{n+1}) - W(t_n))$$

# Stochastic Integral

To integrate

$$\phi_t = \int_0^t f(s, \omega)\, dW_t$$

over $[0, t]$ take partial sums.

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$$\lim_{N\to\infty} \phi_N(\alpha)_t = \lim_{N\to\infty} \sum_{i=1}^{N} f(t_i^\alpha, \omega)(W_{t_i} - W_{t_{i-1}}),$$

$W_t$ is the Wiener process.

$$t_i^\alpha = \alpha t_i + (1-\alpha) t_{i-1}, \qquad \text{for all } 0 \le \alpha \le 1$$

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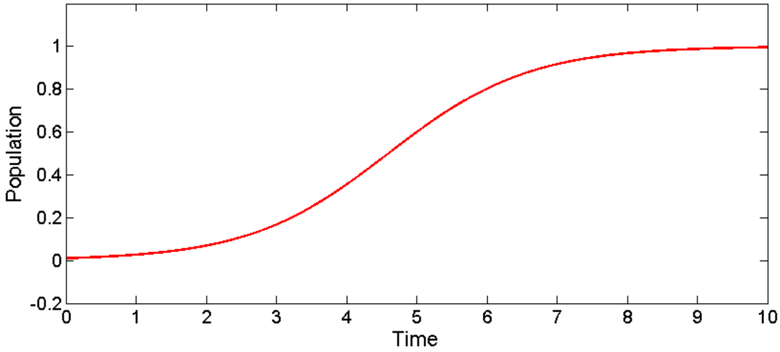
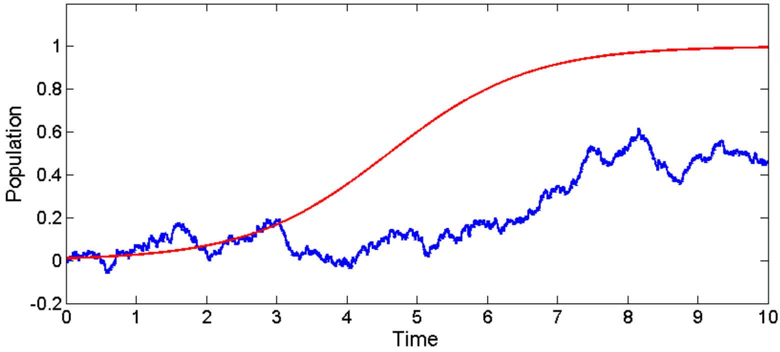
$$\int_0^t W_s\, d_\alpha W_s, = \frac{1}{2}W_t^2 - \left(\frac{1}{2} - \alpha\right) t$$

Special cases $\alpha = 0$ Itô, $\alpha = 1/2$ Stratonovich, $\alpha = 1$ anti-Itô.

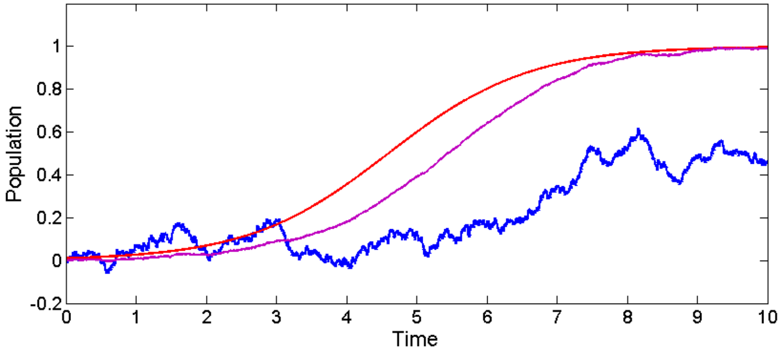



Population growth.




Population growth and Brownian motion.

$E[W(t_n) - W(t_{n-1})] = 0$




Population growth with random fluctuations. $E\left[\frac{dP}{dt}\right] = b(P)$.

# Kramers

- With physical systems assume that noise is uncorrelated.
- With better time sampling, see noise is correlation on a short time scale
- For SDE,

$$dP(t) = P(t)(1 - P(t))\,dt + \sigma dW_t$$

  Population is not differentiable.
- For particle, acceleration is not well defined.

# Colored Noise

- Instead, convolve $\eta(t)$ with a smooth function $g$

$$\eta_\epsilon(t) = \frac{1}{\epsilon}\eta(t) * g\left(\frac{t}{\epsilon^2}\right) \to \eta(t) \text{ as } \epsilon \to 0$$

- $\eta_\epsilon$ is called colored noise.

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$$dx(t) = b(x(t), t) + \frac{\sigma(x(t), t)}{\epsilon} g\left(\frac{t}{\epsilon^2}\right) * \eta(t)\, dt$$

- For $b$ smooth, $x(t)$ has the smoothness properties of $g(t)$. (e.g. $g \in C^\infty \implies x(t) \in C^\infty$.)
- This coincides with our view of nature, particles have a well defined acceleration and velocity.

# O-U Colored Noise

- Ornstein-Ulhenbeck SP, important for computations,

$$dy(t) = -\frac{ay(t)}{\epsilon^2}\,dt + \sqrt{\frac{2a}{\epsilon^2}}dW(t), \quad y_0 = 0.$$

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$$y(t) = \sqrt{\frac{2a}{\epsilon^2}} \int_0^t \exp\left\{ \frac{a}{\epsilon^2}(s-t) \right\}\, dW(s)$$

- Mean zero Gaussian random variable with covariance

$$E[y(t_1)y(t_2)] = \exp\left\{ -\frac{a}{\epsilon^2} |t_1 - t_2| \right\}.$$

# System of ODE's

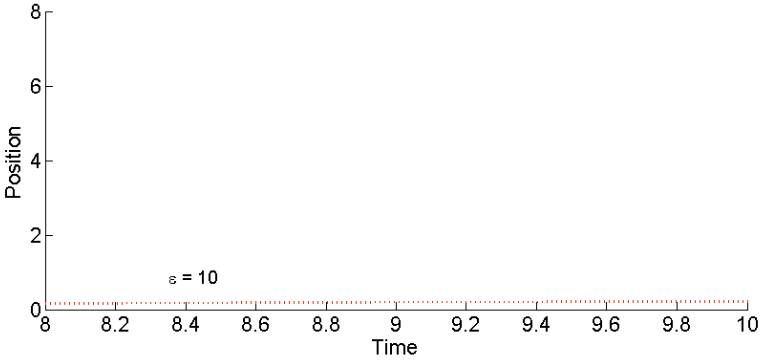
Using OU colored noise we have system of the form,

$$dx(t) = b(x(t)) + \frac{\sigma(x(t))y(t)}{\epsilon}\,dt$$
$$dy(t) = -\frac{ay(t)}{\epsilon^2}\,dt + \sqrt{\frac{2a}{\epsilon^2}}\,dW(t).$$

Interested as $\epsilon \to 0$. Expect to converge to the equation,

$$dX(t) = b(X(t))\,dt + \sqrt{2}\sigma(X(t))\,dW_t$$

# Pictures of results



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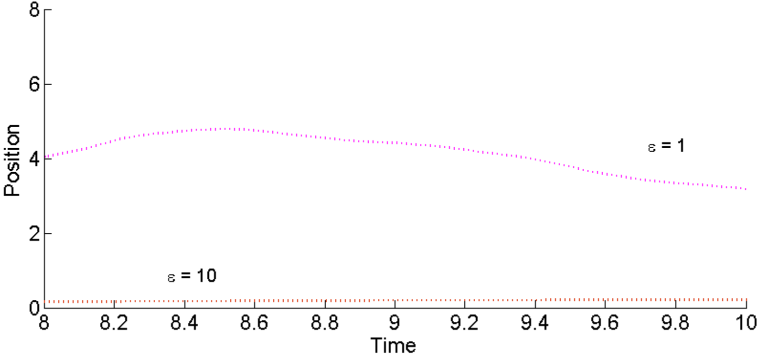

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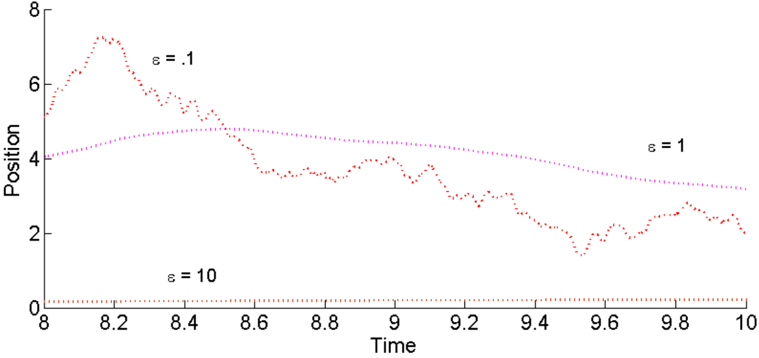

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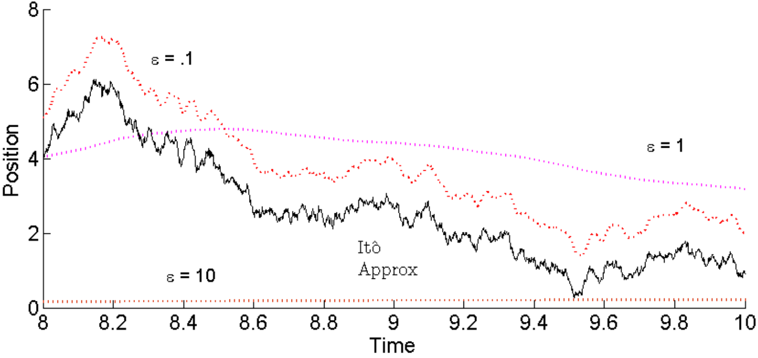

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- Well known, for colored noise approximation, converge to

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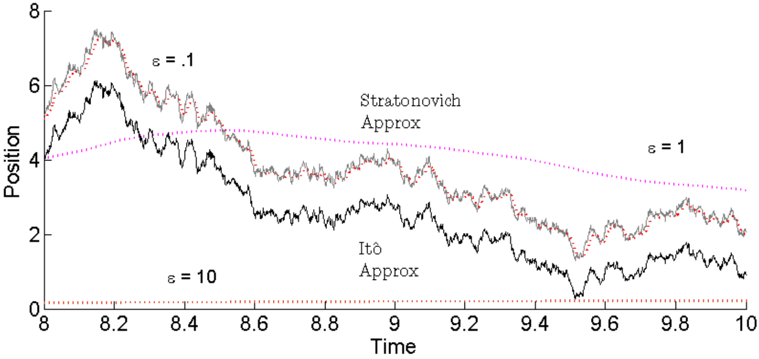
- Well known, for colored noise approximation, converge to

$$dX(t) = b(X(t))\,dt + \sqrt{2}\sigma(X(t))\,d_{\alpha=.5}W_t$$

# Pictures of results



# Which integral do we take?

- Density $p(x', y', t'|x, y, t)$ satisfies the *Backward Kolmogorov* equation

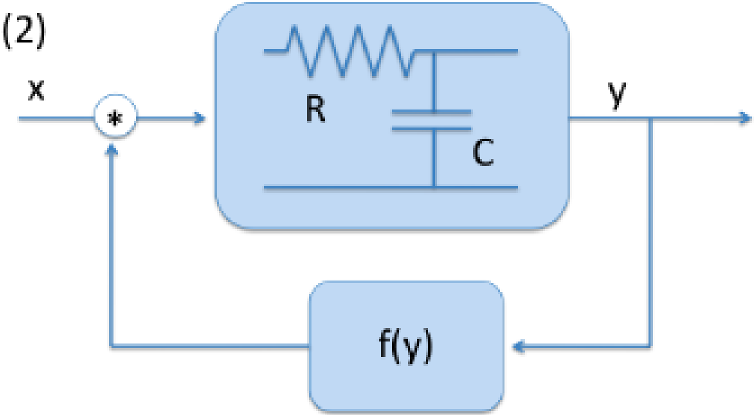
$$\begin{aligned}\frac{\partial p}{\partial t} &= \left(b(x) + \frac{\sigma(x)y}{\epsilon}\right)\frac{\partial p}{\partial x} + \left(\frac{-ay}{\epsilon^2}\right)\frac{\partial p}{\partial v} + \frac{a}{\epsilon^2}\frac{\partial^2 p}{\partial y^2} \\ &= L^{\epsilon}_{x,y} p,\end{aligned}$$

- Use Homogenization theory to find a limiting operator,

$$L^{\epsilon}_{x,y} \to L_x = \sigma(x)^2 \frac{\partial^2}{\partial x^2} + (b(x) + \sigma(x)\sigma'(x))\frac{\partial}{\partial x}$$

- Analysis leads to convergence of distributions of processes $(x(t) \Rightarrow X(t))$.

# Circuit Experiment




(Courtesy G. Volpe)

# Equations and parameters

- Stochastic delay differential equation

$$dx(t) = \frac{-x(t)}{RC} + \frac{D(x(t-d))y(t)}{\epsilon RC}\, dt,$$

- $y(t)$ is colored noise with correlation time $\epsilon^2$ and $d$ is the delay of the feedback.

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- $y(t)$ is colored noise with correlation time $\epsilon^2$ and $d$ is the delay of the feedback.
- First goal is to consider limiting cases of $\epsilon, \delta \to 0$.
- Converge to,

$$dX(t) = \frac{-X(t)}{RC} + \frac{D(X(t))}{RC}\; d_\alpha W_t,$$

- what is $\alpha$?

- For $d \ll \epsilon$, then take $d = 0$,

$$dx(t) = \frac{-x(t)}{RC} + \frac{D(x(t))y(t)}{\epsilon RC}\ dt$$

Converges to

$$\begin{aligned} dX(t) &= \frac{-X(t)}{RC} + \frac{1}{2}D'(X(t))D(X(t))dt + \frac{D(X(t))}{RC}\ dW_t \\ &= \frac{-X(t)}{RC}dt + \frac{D(X(t))}{RC}\ d_{.5}W_t. \end{aligned}$$

- For $\epsilon \ll d$,

$$dX(t) = \frac{-X(t)}{RC}dt + \frac{D(X(t-d))}{RC}\, d_\alpha W_t$$

- Numerically

$$X(t_{n+1}) = X(t_n) + \frac{-X(t_n)}{RC}(t_{n+1} - t_n) + \frac{D(X(t_n^\alpha - d))}{RC}\,(W_{n+1} - W_n)$$

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- For $d = 0$, $\alpha = 1$

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- Need to solve implicitly.

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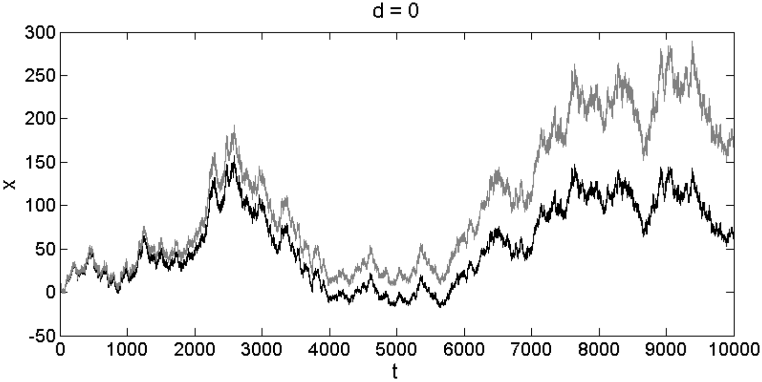
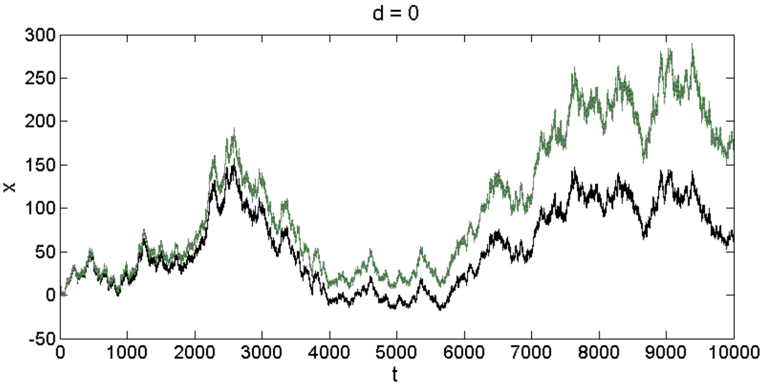
- Need to solve implicitly.
- For $d = \Delta t$,

$$X(t_{n+1}) = X(t_n) + \frac{-X(t_n)}{RC}(t_{n+1} - t_n) + \frac{D(X(t_{n+1} - d))}{RC} \, (W_{n+1} - W_n)$$

- Solves explicitly, like $\alpha = 0$.




Gray = Stratonovich ($\alpha = .5$), Black = Itô ($\alpha = 0$)

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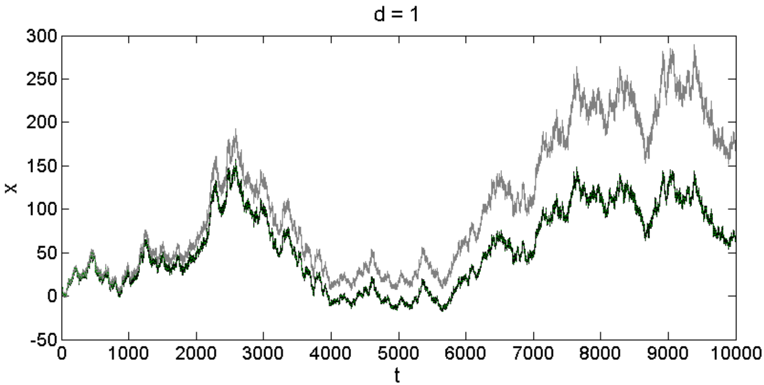


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# Intermediate Cases

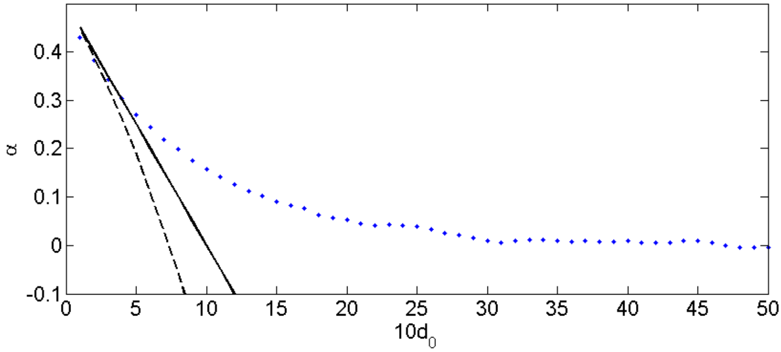
- Now $d \approx \epsilon^2$, set $d = d_0\epsilon^2$ and Taylor expand,

$$dx(t) = \frac{-x(t)}{RC} + \frac{(D(x(t)) - dD'(x(t)))y(t)}{\epsilon RC}\,dt$$
$$dy(t) = -\frac{ay(t)}{\epsilon^2}\,dt + \sqrt{\frac{2a}{\epsilon^2}}\,dW(t).$$

Take first and second order approximations, solve for $\alpha$ in

$$dx(t) = \frac{-x(t)}{RC} + \frac{2\alpha}{a}D'(x(t))D(x(t)) + \sqrt{\frac{2}{a}}D(x(t))\,dW(t)$$

- Also, use an iteration method with original equation to approximate $\alpha$

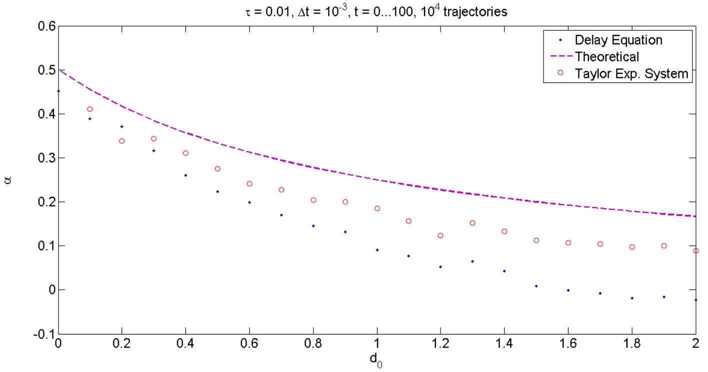



Black solid = 1st order, Black dotted = 2nd order, Blue dots = iteration scheme.

- Instead make a change in time $u = t + d$, then Taylor expand,

$$\dot{x}(u) + d\ddot{x}(u) = -\frac{x(u)}{RC} - \frac{d\dot{x}(u)}{RC} + \frac{D(x(u))}{RC}\frac{1}{\epsilon}y(u+d)$$
$$dy(u+d) = -\frac{\lambda}{\epsilon^2}y(u+d)\,dt + \frac{\sqrt{2\lambda}}{\epsilon}dW(u+d).$$

- $y$ is invariant under shifts, so define $\hat{y}(u) = y(u+d)$, to be another OU process.


τ = 0.01, Δt = 10^-3, t = 0...100, 10^4 trajectories
Delay Equation
Theoretical
Taylor Exp. System
α
d0
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

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- Austin has solution. Colored noise approximation written as 2-D SDE.
- Where are the errors happening?