

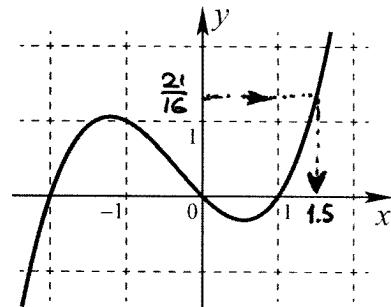
Solution,

1. Consider the function $y(x)$ graphed at the right.

(a) The function $y(x)$ is not invertible. Why?

Some values of y correspond to several values of x , e.g., $y(-2) = y(0) = y(1) = 0$, so $y^{-1}(0)$ is undefined.

(b) When the domain of $y(x)$ is restricted to $x \geq 1$, the function becomes invertible. Estimate the value of the inverse function $y^{-1}(21/16)$.



$$y^{-1}\left(\frac{21}{16}\right) \approx 1.5$$

such value of x that $y(x) = \frac{21}{16}$

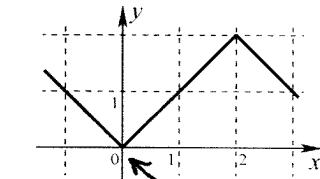
$g(0)=0, g'(0)=1, g'(2)$ is not defined

2. Consider a function $f(x) = \sqrt{g^2(x) + 10^{-6}}$, where $g(x) = 2 - |x - 2|$. (You don't need to explain your reasoning here, when asked about whether the function is differentiable or not.)

(a) Is $f(x)$ differentiable at $x = 0$? If yes, find $f'(0)$.

$$f'(x) = \frac{1}{2\sqrt{g^2(x)+10^{-6}}} \cdot 2g(x)g'(x)$$

$$f'(0) = \frac{g(0)g'(0)}{\sqrt{g^2(0)+10^{-6}}} = \frac{0 \cdot 1}{\sqrt{10^{-6}}} = \boxed{0}$$



yes there is actually no sharp corner here

(b) Is $f(x)$ differentiable at $x = 2$? If yes, find $f'(2)$.

$g'(2)$ does not exist (or $g(x)$ is not differentiable at $x=2$), thus $f(x)$ is not differentiable at $x=2$ too.

no

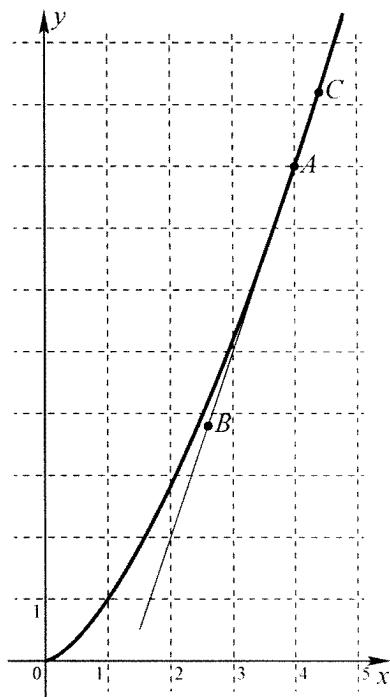
3. Consider a tangent line to the graph of the function $f(x) = x^{3/2}$ at the point $x = 4$.

(a) What is $f(4)$? What is $f'(4)$?

$$f(4) = 4^{3/2} = \boxed{8}$$

$$f'(x) = \frac{3}{2} \sqrt{x}$$

$$f'(4) = \frac{3}{2} \sqrt{4} = \boxed{3}$$



(b) Find the coordinates of the points $A = (4, ?)$, $B = (2.6, ?)$, $C = (4.4, ?)$ that lie on the tangent line (i.e., estimate $f(2.6)$ and $f(4.4)$ from the approximation by the tangent line).

tangent line: $y = f(4) + f'(4)(x - 4) = 8 + 3(x - 4) = 3x - 4$

$$A = (4, 3 \cdot 4 - 4) = \boxed{(4, 8)}$$

$$B = (2.6, 3 \cdot 2.6 - 4) = \boxed{(2.6, 3.8)}$$

$$C = (4.4, 3 \cdot 4.4 - 4) = \boxed{(4.4, 9.2)}$$

4. Find whether the following limits exist and if they do exist, find their values:

(a) $\lim_{x \rightarrow 0} \frac{\sin x}{\log_{10}(1+2x)}$ @ $x = 0$... $\frac{\sin x}{\log_{10}(1+2x)} = \frac{0}{0}$

we have $\frac{0}{0}$ uncertainty,
let us use l'Hôpital's rule

$$\lim_{x \rightarrow 0} \frac{(\sin x)'}{(\log_{10}(1+2x))'} = \lim_{x \rightarrow 0} \frac{\cos x}{\frac{1}{\ln 10} \cdot \frac{1}{1+2x} \cdot 2} = \boxed{\frac{\ln 10}{2}}$$

(b) $\lim_{x \rightarrow +\infty} \sin x$ does not exist

We have $\sin(\pi n) = 0$ and $\sin(\frac{\pi}{2} + 2\pi n) = 1$ for any integer n . No matter how large x is, you can further increase it and end up on points of both types, and $0 \neq 1$.

$$(c) \lim_{x \rightarrow 1^-} \frac{(1-x^2)^{2/3}}{\sin(\pi x)} @ x=1 \quad \frac{(1-x^2)^{2/3}}{\sin \pi x} = \frac{0^{2/3}}{0} = 0$$

||

we have $\frac{0}{0}$ uncertainty
let us use l'Hôpital's rule

$$\lim_{x \rightarrow 1^-} \frac{\frac{2}{3}(1-x^2)^{-1/3}(-2x)}{\pi \cos \pi x} = \lim_{x \rightarrow 1^-} \frac{-4x}{3\pi \cos \pi x} \cdot \frac{1}{(1-x^2)^{1/3}}$$

$\downarrow x \rightarrow 1^-$ $\downarrow x \rightarrow 1^-$
 $\frac{4}{3\pi}$ $+\infty$

does not exist

5. Find the derivatives of the following functions. Assume that a is a constant.

$$(a) f(x) = x^3 + x^{-3}$$

$$f'(x) = 3x^{3-1} + (-3)x^{-3-1} = \boxed{3x^2 - \frac{3}{x^4}}$$

$$(b) g(t) = 7^t - 7^a$$

$$g'(t) = (7^t)' - \cancel{(7^a)'} = \boxed{7^t \ln 7}$$

-OR-

$$(7^t)' = ((e^{\ln 7})^t)' \stackrel{O, a \text{ is a constant}}{=} (e^{t \cdot \ln 7})' = e^{t \cdot \ln 7} \cdot (\ln 7)$$

$$(c) x(h) = \sqrt{1+e^{-ah}}$$

$$x'(h) = \frac{1}{2\sqrt{1+e^{-ah}}} (1+e^{-ah})' = \boxed{\frac{-ae^{-ah}}{2\sqrt{1+e^{-ah}}}}$$

6. The number of bacterias in a colony is growing exponentially with a doubling time being equal to 3 hours. At 11:00am the number of bacterias was equal to 10^6 .

- (a) At what rate the number of bacterias was increasing at 11:00pm (the same day)? Give units.

$$N(t) = \frac{10^6 \cdot e^{rt}}{N @ 10\text{am}} \stackrel{\text{counted in hours from 11am}}{=} 10^6 \cdot 2^{\frac{t}{3}}$$

$e^{\frac{r \cdot 3}{3}} = 2 \Rightarrow r = \frac{1}{3} \ln 2$

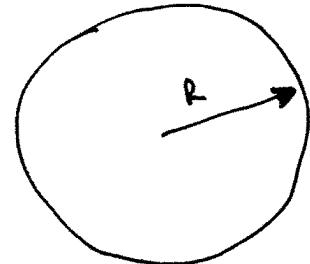
3 hours is doubling time

doubling in three hours
if t is increased by 3, then $\frac{t}{3}$ is increased by 1, and N is multiplied by 2

$$\frac{dN}{dt} = 10^6 \cdot r \cdot e^{rt}, \quad \left. \frac{dN}{dt} \right|_{t=12} = \boxed{10^6 \frac{\ln 2}{3} \cdot 16 \frac{1}{\text{hours}}}$$

12 hours from 11am is 11pm

- (b) The bacterias form a circular spot in a Petri dish, whose area is the number of bacterias multiplied by $20 \mu\text{m}^2$. At what rate the radius of the spot was increasing at 11:00pm?



$$\pi R^2(t) = N(t) \cdot 20 \mu\text{m}^2$$

$$R(t) = \sqrt{\frac{20 \mu\text{m}^2}{\pi}} N(t)$$

$$\frac{dR}{dt} = \frac{1}{2\sqrt{\frac{20 \mu\text{m}^2}{\pi} \cdot N(t)}} \cdot \frac{20 \mu\text{m}^2}{\pi} \frac{dN}{dt}$$

at $t=12$ or 11pm

$$= \frac{1}{2} \sqrt{\frac{20}{\pi}} \mu\text{m} \cdot \frac{1}{\sqrt{16 \cdot 10^6}} \cdot 10^6 \frac{\ln 2}{3} \cdot 16 \frac{1}{\text{hour}} =$$

$$= \boxed{\frac{5}{\pi} \frac{4 \cdot 10^3 \cdot \ln 2}{3} \frac{\mu\text{m}}{\text{hour}}}$$

- O R -

$$R(t) = \sqrt{\frac{20}{\pi}} 10^3 e^{\frac{rt}{2}} \mu\text{m}$$

$$\frac{dR}{dt} = \sqrt{\frac{20}{\pi}} 10^3 \frac{r}{2} e^{\frac{rt}{2}} \frac{\mu\text{m}}{\text{hour}}$$

@ $t=12$

7. Find the exact global maximum and minimum values of the function $f(x) = \sin(x) + x/2$ on the interval $[0, 10]$.

The point x_* at which the function $f(x)$ has its global maximum or minimum is either critical point (i.e., $f'(x_*)=0$ or $f'(x_*)$ is undefined) or an endpoint of the interval.

$f(x)$ is everywhere differentiable, and $f'(x) = \cos x + \frac{1}{2}$.

We have $f'(x)=0$ if $\cos x = -\frac{1}{2}$

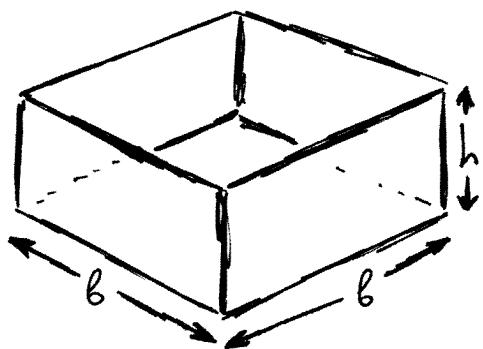
$$\text{other values} \rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ or } \frac{8\pi}{3}$$

outside the interval $[-\arccos(-\frac{1}{2}), \arccos(-\frac{1}{2})]$ are

Thus, we need to check 5 points:

x	$f(x)$	
global minimum @ $x = 0$	0	0 - smallest
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2} + \frac{\pi}{3}$	1.913
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2} + \frac{2\pi}{3}$	1.228
global maximum @ $x = \frac{8\pi}{3}$	$\frac{\sqrt{3}}{2} + \frac{4\pi}{3}$	5.055 - largest
10	$\sin 10 + 5$	4.456

8. A cardboard box with square base and with no top has a fixed volume V . What dimensions minimize the cost of the box? (The cost is proportional to the box's area, i.e., area of 4 sides plus the area of the base.)



$$V = b^2 h$$

$$\text{area} = \frac{b^2}{\text{base}} + \frac{4bh}{4 \text{sides}}$$

$$h = \frac{V}{b^2} \Rightarrow \text{area}(b) = b^2 + 4b \frac{V}{b^2} = b^2 + \frac{4V}{b}$$

$$\frac{d\text{area}}{db} = 2b - \frac{4V}{b^2} \stackrel{\text{in the minimum}}{=} 0$$

$$\frac{d^2\text{area}}{db^2} = 2 + \frac{8V}{b^3} \stackrel{\text{minimum}}{>} 0$$

$$b^3 = 2V$$

$$\boxed{b = (2V)^{1/3}}$$

$$\boxed{h = \left(\frac{V}{4}\right)^{1/3}}$$

- O.R. -

$$b = \sqrt{\frac{V}{h}} \Rightarrow \text{area}(h) = \frac{V}{h} + 4\sqrt{\frac{V}{h}} \cdot h = \frac{V}{h} + 4\sqrt{Vh}$$

$$\frac{d\text{area}}{dh} = -\frac{V}{h^2} + 2\sqrt{\frac{V}{h}} \stackrel{\text{in the minimum}}{=} 0$$

$$\frac{V}{h^2} = 2\sqrt{\frac{V}{h}} \Rightarrow \frac{V^2}{h^4} = 4\frac{V}{h} \Rightarrow h^3 = \frac{V}{4}$$

$$\frac{d^2\text{area}}{dh^2} = \frac{2V}{h^3} - \frac{\sqrt{V}}{h^{3/2}} = 8 - 2 = 6 > 0$$

minimum