

1. Split the function $\frac{2(s+5)}{(s-1)^2(s^2+3)}$ into partial fractions.

$$\frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{Cs+D}{s^2+3}$$

$$\frac{3}{(s-1)^2} - \frac{1}{s-1} + \frac{s-2}{s^2+3}$$

$$2(s+5) = A(s^2+3) + B\underbrace{(s-1)(s^2+3)}_{s^3-s^2+3s-3} + (Cs+D)\underbrace{(s-1)^2}_{s^2-2s+1} =$$

$$= (B+C)s^3 + (A-B-2C+D)s^2 + \\ + (3B+C-2D)s + (3A-3B+D)$$

$$B+C=0 \Rightarrow C=-B$$

$$A-B-2(-B)+D=0 \Rightarrow D=-A-B$$

$$3B+(-B)-2(-A-B)=2 \Rightarrow \begin{cases} 2A+4B=2 \\ 2A-4B=10 \end{cases}$$

$$3A-3B+(-A-B)=10 \Rightarrow \begin{cases} 2A+4B=2 \\ 2A-4B=10 \end{cases}$$

$$\frac{2A+4B}{2} + \frac{2A-4B}{10} = 12$$

$$A=3$$

$$B = \frac{2-2A}{4} = \frac{2-6}{4} = -1$$

$$C = -B = 1$$

$$D = -A - B = -3 + 1 = -2$$

1. Split the function $\frac{3(y^2+y)}{(y^2+1)(y^2+4)}$ into partial fractions.

$$\frac{Ay+B}{y^2+1} + \frac{Cy+D}{y^2+4}$$

$$\frac{y-1}{y^2+1} + \frac{-y+4}{y^2+4}$$

$$3(y^2+y) = (Ay+B)(y^2+4) + (Cy+D)(y^2+1) = \\ = (\underbrace{A+C}_0)y^3 + (\underbrace{B+D}_3)y^2 + (\underbrace{4A+C}_3)y + (\underbrace{4B+D}_0)$$

$$\begin{cases} A+C=0 \Rightarrow C=-A \\ 4A+C=3 \Rightarrow 3A=3 \end{cases}, \quad \boxed{\begin{array}{l} A=1 \\ C=-1 \end{array}}$$

$$\begin{cases} B+D=3 \\ 4B+D=0 \Rightarrow D=-4B \end{cases}$$

$$B+(-4B) = -3B = 3$$

$$\boxed{\begin{array}{l} B=-1 \\ D=4 \end{array}}$$

$$\begin{aligned}
 2. \text{ Find the integral } \int z^2 \ln(2z+1) dz &= \left\{ \begin{array}{l} u = 2z+1 \\ z = \frac{u-1}{2} \\ dz = du/2 \end{array} \right\} = \\
 &= \int \left(\frac{u-1}{2}\right)^2 \ln u \frac{du}{2} = \frac{1}{8} \int (u^2 - 2u + 1) \ln u \, du = \\
 &= \frac{1}{8} \underbrace{\int u^2 \ln u \, du}_{\substack{\text{Table III-13, } n=2 \\ \frac{u^3}{3} \ln u - \frac{u^3}{9}}} - \frac{1}{4} \underbrace{\int u \ln u \, du}_{\substack{\text{Table III-13, } n=1 \\ \frac{u^2}{2} \ln u - \frac{u^2}{4}}} + \frac{1}{8} \underbrace{\int \ln u \, du}_{\substack{\text{Table I-4 or} \\ \text{Table III-13, } n=0 \\ u \ln u - u}} = \\
 &= \frac{u^3}{24} \ln u - \frac{u^3}{72} - \frac{u^2}{8} \ln u + \frac{u^2}{16} + \frac{u}{8} \ln u - \frac{u}{8} + C = \\
 &= \boxed{\left(\frac{(2z+1)^3}{24} - \frac{(2z+1)^2}{8} + \frac{2z+1}{8} \right) \ln(2z+1) - \frac{(2z+1)^3}{72} + \frac{(2z+1)^2}{16} - \frac{2z+1}{8} + C} \\
 &\quad \text{||} \\
 &= \left(\frac{z^3}{3} + \frac{1}{24} \right) \ln(2z+1) - \frac{z^3}{9} + \frac{z^2}{12} - \frac{z}{12} + C'
 \end{aligned}$$

- OR -

$$\begin{aligned}
 \int dz \underbrace{\frac{z^2}{u'} \ln(2z+1)}_{\substack{\text{u' } \\ \text{z}}} &= \left\{ \begin{array}{l} u = \frac{z^3}{3}, \quad u' = \frac{z^2}{2z+1} \end{array} \right\} = \\
 &= \frac{z^3}{3} \ln(2z+1) - \frac{1}{3} \int \frac{dz}{z+\frac{1}{2}} \underbrace{z^3}_{\substack{\text{||}}} = \\
 &= \boxed{\frac{z^3}{3} \ln(2z+1) - \frac{1}{9} \left(z+\frac{1}{2}\right)^3 + \frac{1}{4} \left(z+\frac{1}{2}\right)^2 - \frac{1}{4} z + \frac{1}{24} \ln\left(z+\frac{1}{2}\right) + \tilde{C}}
 \end{aligned}$$

3. Consider an integral $\int_0^1 x^3 dx$. optional, was not asked for

$$= \frac{x^4}{4} \Big|_0^1 = \frac{1}{4} = 0.25$$

Identify each approximation as overestimate or an underestimate.

approximation	over	under	x^3 is	increasing	decreasing	concave up	concave down
LEFT(12)		✓	why?	✓			
RIGHT(3)	✓		why?	✓			
TRAP(5)	✓		why?			✓	
MID(7)		✓	why?			✓	

Calculate the following approximation to the integral.

approximation	
LEFT(5)	$\frac{0.2}{\Delta x} \cdot (0^3 + 0.2^3 + 0.4^3 + 0.6^3 + 0.8^3) =$ $= 0.2 (0.008 + 0.064 + 0.216 + 0.512) =$ $= 0.2 \cdot 0.8 = \boxed{0.16}$
RIGHT(5)	$\frac{0.2}{\Delta x} \cdot (0.2^3 + 0.4^3 + 0.6^3 + 0.8^3 + 1^3) =$ <p style="text-align: center; margin-left: 100px;">would generate LEFT(5)</p> $= \text{LEFT}(5) + 0.2 = \boxed{0.36}$
TRAP(5)	$\frac{1}{2} (\text{LEFT}(5) + \text{RIGHT}(5)) = \boxed{0.26}$ <p style="text-align: center; margin-left: 100px;">- OR -</p> $\frac{0.2}{\Delta x} \cdot \left(\frac{1}{2} \cdot 0^3 + 0.2^3 + 0.4^3 + 0.6^3 + 0.8^3 + \frac{1}{2} \cdot 1^3\right) = 0.2 \cdot 1.3$
MID(5)	$\frac{0.2}{\Delta x} \cdot (0.1^3 + 0.3^3 + 0.5^3 + 0.7^3 + 0.9^3) =$ $= 0.2 \cdot (0.001 + 0.027 + 0.125 + 0.343 + 0.729) =$ $= 0.2 \cdot 1.225 = \boxed{0.245}$
SIMP(5)	$\frac{2 \cdot \text{MID}(5) + \text{TRAP}(5)}{3} = \frac{0.49 + 0.26}{3} = \boxed{0.25}$ <p style="text-align: center; margin-left: 100px;">Simpson's rule integrates polynomials up to the 3rd degree exactly</p>

4. Integrate the function $\frac{1}{\sqrt{8+2x-x^2}}$ with respect to x .

$$\begin{aligned} \int \frac{dx}{\sqrt{8+2x-x^2}} &= \int \frac{dx}{\sqrt{8+\underbrace{1-(x-1)^2}_{2x-x^2}}} = \int \frac{dx}{\sqrt{3^2-(x-1)^2}} = \\ &= \left\{ \begin{array}{l} u = x-1 \\ du = dx \end{array} \right\} = \int \frac{du}{\sqrt{3^2-u^2}} = \left\{ \begin{array}{l} \text{Table VI-28} \\ a=3 \end{array} \right\} = \\ &= \arcsin \frac{u}{3} + C = \boxed{\arcsin \frac{x-1}{3} + C} \end{aligned}$$

or use

$u = 3 \sin \theta$
substitution

5. Calculate the integral $\int_4^\infty \frac{dx}{x^2-1}$, if it converges.

$$\begin{aligned} \int_4^\infty \frac{dx}{(x-1)(x+1)} &= \left\{ \begin{array}{l} \text{Table V-26} \\ a=1, b=-1 \end{array} \right\} = \lim_{R \rightarrow +\infty} \frac{1}{1-(-1)} \ln \left| \frac{x-1}{x-(-1)} \right| \Big|_4^R = \\ &\quad \text{||-partial fractions} \\ \int_4^\infty dx \left(\frac{1}{2} \frac{1}{x-1} - \frac{1}{2} \frac{1}{x+1} \right) &= \\ &= \frac{1}{2} \lim_{R \rightarrow \infty} \ln \left(\frac{R-1}{R+1} \right) - \frac{1}{2} \ln \frac{4-1}{4+1} = \boxed{\frac{1}{2} \ln \frac{5}{3}} \end{aligned}$$

6. Decide if the improper integral converges or diverges.

$$\int_1^{\infty} \frac{dy}{1+\sqrt{y}} \quad \boxed{\text{diverges}} \leftarrow \begin{cases} \frac{1}{1+\sqrt{y}} \geq \frac{1}{2\sqrt{y}} & \text{for } y \geq 1 \\ \int_1^{\infty} \frac{dy}{2y^{1/2}} \quad \text{diverges} & - \text{p-test, } p = \frac{1}{2} \leq 1 \end{cases} - \text{comparison test}$$

$$\int_{-\infty}^0 e^x dx = \lim_{A \rightarrow -\infty} \int_A^0 e^x dx = \lim_{A \rightarrow -\infty} e^x \Big|_A^0 = \lim_{A \rightarrow -\infty} (1 - e^A) = 1$$

Converges

$$\int_0^{\pi} \frac{d\theta}{\sin \theta} = \left\{ \text{Table IV-20} \right\} = \lim_{\substack{\delta \rightarrow 0^+ \\ \epsilon \rightarrow 0^+}} \frac{1}{2} \ln \left| \frac{1 - \cos \theta}{1 + \cos \theta} \right| \Big|_{\delta}^{\pi - \epsilon} \quad \boxed{\text{diverges}}$$

- OR -

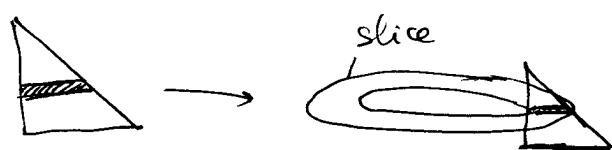
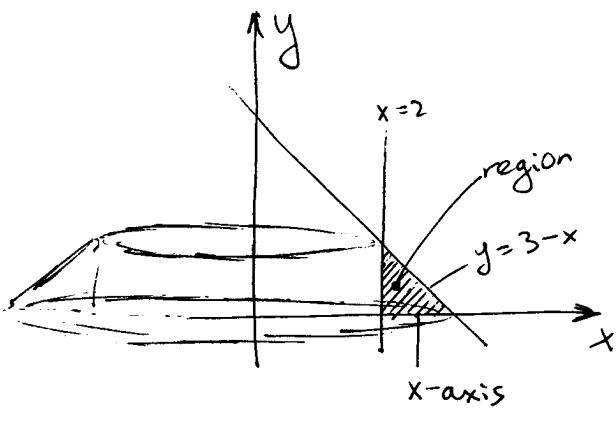
$$\boxed{\text{diverges}} \leftarrow \begin{cases} \frac{1}{\sin \theta} > \frac{1}{\theta} & \text{for } \theta > 0 \\ \int_0^{\pi} \frac{d\theta}{\theta} \quad \text{diverges} & - \text{p-test, } p = 1 \end{cases} - \text{comparison test}$$

$$\int_{-1}^0 \frac{e^t dt}{(e^t - 1)^2} = \left\{ \begin{array}{l} u = e^t - 1 \\ du = e^t dt \end{array} \right\} = \int_{\frac{1}{e}-1}^0 \frac{du}{u^2} \quad \boxed{\text{diverges}}$$

p-test, $p = 2 \geq 1$

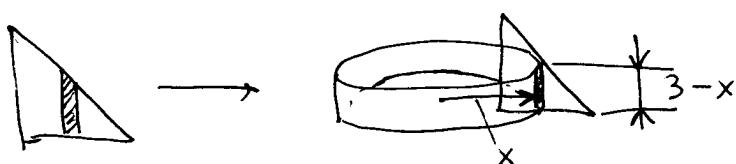
$$\int_0^1 \frac{dx}{\sqrt{x+x^2}} \quad \boxed{\text{Converges}} \leftarrow \begin{cases} \frac{1}{\sqrt{x+x^2}} < \frac{1}{\sqrt{x}} & - \text{comparison test} \\ \int_0^1 \frac{dx}{x^{1/2}} \quad \text{converges} & - \text{p-test, } p = \frac{1}{2} < 1 \end{cases}$$

7. A solid is obtained by rotating a region, that is bounded by $x = 2$, $y = 3 - x$ and the x -axis, around the y -axis. Find its volume.



$$\begin{aligned} \text{volume} &= \int_0^1 dy \pi ((3-y)^2 - 2^2) = \\ &\quad r_{\text{out}}(y) = 3-y, \quad r_{\text{in}}(y) = 2 \\ &= \pi \int_0^1 dy (5 - 6y + y^2) = \pi (5 - 3 + \frac{1}{3}) = \boxed{\frac{7\pi}{3}} \end{aligned}$$

- OR -



$$\begin{aligned} \text{volume} &= \int_2^3 dx 2\pi x \cdot (3-x) = \\ &= 2\pi \left(\frac{3x^2}{2} - \frac{x^3}{3} \right) \Big|_2^3 = 2\pi \left(\frac{27}{2} - \frac{27}{3} - 6 + \frac{8}{3} \right) = \\ &= \frac{2\pi}{6} (81 - 54 - 36 + 16) = \boxed{\frac{7\pi}{3}} \end{aligned}$$