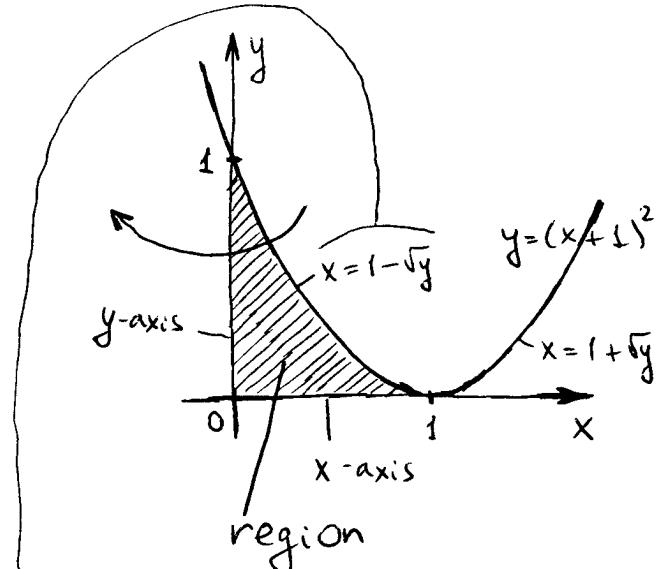
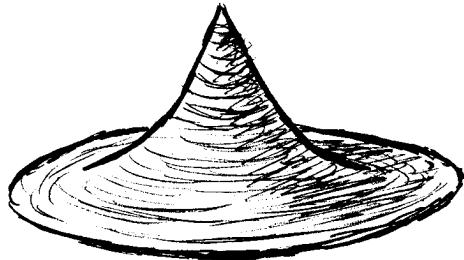


1. A solid with uniform density is obtained by rotating around the y -axis the region bounded by the x -axis, the y -axis, and by $y = (x - 1)^2$. (a) Sketch the solid. (b) Find its volume. (c) Find \bar{y} , i.e., y -coordinate of the center of mass.

(a)



(b)

$$\begin{aligned} \text{Volume} &= \int_0^1 dy \pi (1 - \sqrt{y})^2 = \\ &= \pi \int_0^1 dy (1 - 2\sqrt{y} + y) = \pi \left(1 - \frac{2}{3} + \frac{1}{2} \right) = \\ &= \pi \left(\frac{3}{2} - \frac{4}{3} \right) = \boxed{\pi/6} \end{aligned}$$

(c)

$$\begin{aligned} \bar{y} &= \frac{\int_0^1 dy \pi (1 - \sqrt{y})^2 \cdot y}{\text{Volume}} = 6 \int_0^1 dy (y - 2y^{3/2} + y^2) = \\ &= 6 \left(\frac{1}{2} - \frac{2}{5} + \frac{1}{3} \right) = \frac{5}{2} - \frac{24}{5} = \boxed{\frac{1}{5}} \end{aligned}$$

2. Do the following sequences converge or diverge? If a sequence converges, find its limit.

(a) $\frac{2^n}{n^8 + 16}$

2^n eventually dominates $n^8 + 16$

diverges

see, e.g., Sec 1.6, p. 46

(b) $\cos(2^{-n})$

$$\xrightarrow{n \rightarrow \infty} \cos\left(\lim_{n \rightarrow \infty} 2^{-n}\right) = \cos(0) = 1$$

$\cos(\cdot)$ is a continuous function

Converges

(c) $\frac{1}{100 + (-1)^n}$

- alternates between $\frac{1}{99}$ and $\frac{1}{101}$

diverges

(d) $a_n \approx \exp(-\sqrt{n})$

$$\xrightarrow{n \rightarrow \infty} 0$$

Converges

Let $\varepsilon > 0$ be an arbitrarily small number.

Consider $N = \lceil (\ln \frac{1}{\varepsilon})^2 \rceil$. $\lceil x \rceil$ is the ceiling function, $\lceil x \rceil$ is the least integer not smaller than x .

For all $n > N$ we have

$$|a_n| < |a_N| = \exp\left(-\sqrt{\lceil (\ln \frac{1}{\varepsilon})^2 \rceil}\right) \leq \exp\left(-\ln \frac{1}{\varepsilon}\right) = \varepsilon.$$

3. Do the following series converge or diverge?

(a) $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$ p-test, $p = \pi > 1$

Converges

(b) $\sum_{k=2}^{\infty} (\ln k)^2$ - terms of the series don't tend to 0 as $k \rightarrow \infty$

Diverges

see, e.g., Sec. 9.3, Theorem 9.2 item 3, p. 507

(c) $\sum_{n=4}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$ sign alternating series
 a_n $0 < a_{n+1} < a_n \xrightarrow[n \rightarrow \infty]{} 0$
 $\frac{1}{\sqrt{n+4}} \quad \frac{1}{\sqrt{n+3}}$

Converges

(d) $\sum_{n=0}^{\infty} \frac{2^n}{3^n}$ geometric series with {constant multiplier} $\frac{2}{3} < 1$
common ratio

- OR -

ratio test

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(2/3)^{n+1}}{(2/3)^n} = \frac{2}{3} < 1$$

Converges

4. Find the radius of convergence of the following power series.

$$(a) \sum_{n=0}^{\infty} \frac{x^n}{5^n + 5^{-n}}$$

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{5^{n+1} + 5^{-n-1}} / \frac{x^n}{5^n + 5^{-n}} \right| = |x| \cdot \lim_{n \rightarrow \infty} \frac{5^n + 5^{-n}}{5^{n+1} + 5^{-n-1}} = \\ &= |x| \cdot \lim_{n \rightarrow \infty} \frac{1 + 5^{-2n}}{5 + 5^{-2n}} = \frac{|x|}{5} \end{aligned}$$

radius of convergence = 5

$$(b) \sum_{m=0}^{\infty} \frac{(x-3)^m}{m^2 + m + 1}$$

$$\begin{aligned} L &= \lim_{m \rightarrow \infty} \left| \frac{(x-3)^{m+1}}{(m+1)^2 + (m+1) + 1} / \frac{(x-3)^m}{m^2 + m + 1} \right| = |x-3| \cdot \lim_{m \rightarrow \infty} \frac{m^2 + m + 1}{m^2 + 3m + 3} = \\ &= |x-3| \lim_{m \rightarrow \infty} \frac{1 + \frac{1}{m} + \frac{1}{m^2}}{1 + \frac{3}{m} + \frac{3}{m^2}} = |x-3| \end{aligned}$$

radius of convergence = 1

5. Find the interval of convergence of the $\sum_{n=1}^{\infty} \frac{(x-1)^n}{3^n \sqrt{n}}$ series.

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{(x-1)^{n+1}}{3^{n+1} \sqrt{n+1}} \right| \cdot \frac{1}{\left| \frac{(x-1)^n}{3^n \sqrt{n}} \right|} = |x-1| \lim_{n \rightarrow \infty} \frac{3^n \sqrt{n}}{3^{n+1} \sqrt{n+1}} = \\ &= |x-1| \cdot \lim_{n \rightarrow \infty} \frac{1}{3 \sqrt{1 + \frac{1}{n}}} = \frac{|x-1|}{3} \text{ - radius of } \\ &\quad \text{convergence } R = 3 \end{aligned}$$

Checking endpoints:

$$x = a - R = -2, \quad \sum_{n=1}^{\infty} \frac{(-2-1)^n}{3^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \text{ - sign alternating series} \quad \underline{\text{converges}}$$

$$x = a + R = 4, \quad \sum_{n=1}^{\infty} \frac{(4-1)^n}{3^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ - p-test, } p = \frac{1}{2} < 1 \quad \underline{\text{diverges}}$$

interval of convergence is $-2 \leq x < 4$

6. Find the approximate value of $1.4^{-1/2}$ using the 3rd degree Taylor polynomial approximating $\frac{1}{\sqrt{1+x}}$ near $x = 0$.

$$\begin{aligned} \frac{1}{\sqrt{1+x}} &= (1+x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}x^3 + \dots \\ &\quad \text{binomial series, } p = -\frac{1}{2} \\ &= 1 - \frac{x}{2} + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots \\ &\quad \text{3rd degree Taylor polynomial} \end{aligned}$$

$$\begin{aligned} \frac{1}{\sqrt{1.4}} &\approx 1 - \frac{0.4}{2} + \frac{3 \cdot 0.4^2}{8} - \frac{5 \cdot 0.4^3}{16} = 0.84 \\ &\quad \parallel \text{use } x = 0.4 \quad \text{exact value} \\ &\quad 0.84515\dots \end{aligned}$$

7. Find the Taylor series for $\cos(\sqrt{1+x} - 1)$ about $x = 0$ up to x^3 (inclusive) terms.

$$\begin{aligned}
 f(x) &= \cos(\sqrt{1+x} - 1) && \text{chain rule} & f(0) &= 1 \\
 f'(x) &= -\sin(\sqrt{1+x} - 1) \cdot \frac{1}{2\sqrt{1+x}} & f'(0) &= 0 \\
 f''(x) &= -\underbrace{\cos(\sqrt{1+x} - 1) \cdot \frac{1}{4(1+x)}}_{\text{product rule}} - \underbrace{\sin(\sqrt{1+x} - 1) \cdot \left(-\frac{1}{4} \frac{1}{(1+x)^{3/2}}\right)}_{\text{product rule}} & f''(0) &= -\frac{1}{4} \\
 f'''(x) &= \cancel{\sin(\dots) \cdot (\dots)} - \cos(\dots) \left(-\frac{1}{4(1+x)^2}\right) - \cos(\dots) \left(-\frac{1}{8(1+x)^2}\right) - \\
 &\quad \cancel{- \sin(\dots) (\dots)} & f'''(0) &= \frac{1}{4} + \frac{1}{8} = \frac{3}{8} \\
 f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \dots &= \boxed{1 - \frac{x^2}{8} + \frac{x^3}{16} + \dots}
 \end{aligned}$$

-OR-

$$\begin{aligned}
 \text{Let } \varepsilon &= \sqrt{1+x} - 1 = \frac{x}{2} - \frac{x^2}{8} + \dots \\
 &\text{binomial series, } p = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \cos(\varepsilon) &= 1 - \frac{\varepsilon^2}{2} + \underbrace{\frac{\varepsilon^4}{24}}_{\text{already too small, like } x^7} + \dots = \\
 &= 1 - \frac{1}{2} \left(\frac{x}{2} - \frac{x^2}{8} + \dots \right)^2 + \dots = \\
 &= \boxed{1 - \frac{x^2}{8} + \frac{x^3}{16} + \dots}
 \end{aligned}$$