We consider a Fredholm integral equation of the 2nd kind

$$u(x) = f(x) + \lambda \int_a^b \mathrm{d}y\, K(x\,|\,y)\, u(y) \tag{1}$$

One of the common ways to solve the eq. (1) is doing the iterations

$$u_{n+1}(x) := f(x) + \lambda \int_a^b \mathrm{d}y\, K(x\,|\,y)\, u_n(y)$$

where u_n is the approximation of u at n^{th} iteration, and $u_0 := f$. If we introduce an operator $\hat{K}$ with the integral kernel $K(x\,|\,y)$, then the eq. (1) can be written as $(\hat{I} - \lambda\hat{K})u = f$, where $\hat{I}$ is the identity operator. The iterations go as $u_{n+1} = f + \lambda\hat{K}u_n$. The approximation at n^{th} iteration is equal to $u_n = \sum_{m=0}^{n} \lambda^m \hat{K}^m f$.

Let us introduce *repeated kernels* $K_n(x\,|\,y)$ — the integral kernels of the operators $\hat{K}^n$. We have $K_0(x\,|\,y) = \delta(x-y)$ as the kernel of the identity operator $\hat{K}^0 = \hat{I}$. Obviously, $K_1(x\,|\,y) = K(x\,|\,y)$, and for any $n \geq 1$ we have

$$K_{n+1}(x\,|\,y) = \int_a^b \mathrm{d}z_1 \int_a^b \mathrm{d}z_2 \cdots \int_a^b \mathrm{d}z_n\; K(x\,|\,z_1)K(z_1\,|\,z_2)\cdots K(z_n\,|\,y)$$

The repeated kernels satisfy the following recurring relation:

$$K_{m+n}(x\,|\,y) = \int_a^b \mathrm{d}z\, K_m(x\,|\,z)K_n(z\,|\,y) \quad , \textit{i.e.,} \quad \hat{K}^{m+n} = \hat{K}^m\hat{K}^n$$

The expansion resulted from the iterations,

$$u(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^b \mathrm{d}y\, K_m(x\,|\,y) f(y)$$

(we did write the $m = 0$ term separately in order not to deal with distributions) may converge (later we will see that it does converge for any λ in the case of a Volterra equation), but could also diverge even if there is a well defined unique solution. A systematic method of obtaining a solution for any λ was suggested by Erik Ivar Fredholm.

Let us discretize the problem, *e.g.*, using a grid of N points $x_i = a + h(i - 1/2)$, $1 \le i \le N$, where $h = (b-a)/N$ is the grid spacing. We denote $f_i := f(x_i)$, $u_i := u(x_i)$, and $K_{ij} := K(x_i\,|\,x_j)$. The integral is approximated by a finite sum (here we use the midpoint rule), so the integral equation is discretized as

$$u_i = f_i + \lambda h \sum_{j=1}^{N} K_{ij} u_j \tag{2}$$

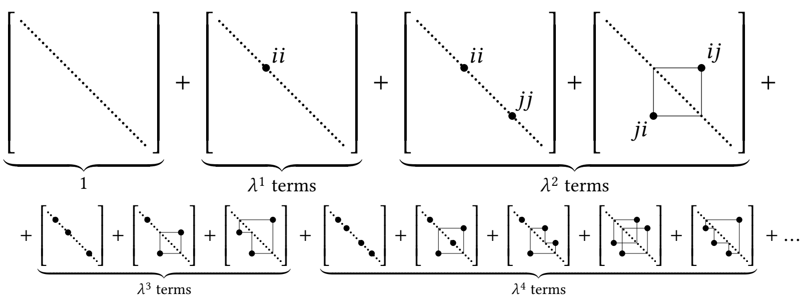
The solution of this system of N linear equations can be written using the Cramer's rule: $u_i = \det \hat{\mathscr{B}}_i / \det \hat{\mathscr{A}}$, where

$$\hat{\mathscr{A}} := \begin{bmatrix} 1-\lambda h K_{11} & -\lambda h K_{12} & -\lambda h K_{13} & \cdots & -\lambda h K_{N-1,1} & -\lambda h K_{N1} \\ -\lambda h K_{21} & 1-\lambda h K_{22} & -\lambda h K_{23} & \cdots & -\lambda h K_{N-1,2} & -\lambda h K_{N2} \\ -\lambda h K_{31} & -\lambda h K_{32} & 1-\lambda h K_{33} & \cdots & -\lambda h K_{N-1,3} & -\lambda h K_{N3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\lambda h K_{N-1,1} & -\lambda h K_{N-1,2} & -\lambda h K_{N-1,3} & \cdots & 1-\lambda h K_{N-1,N-1} & -\lambda h K_{N,N-1} \\ -\lambda h K_{N1} & -\lambda h K_{N2} & -\lambda h K_{N3} & \cdots & -\lambda h K_{N,N-1} & 1-\lambda h K_{NN} \end{bmatrix}$$

is the matrix of the system (2), and the matrix $\hat{\mathscr{B}}_i$ is obtained from the matrix $\hat{\mathscr{A}}$ by substituting the i^{th} column by the column vector $\boldsymbol{f}$.

We are interested in how the solution $u(x)$ depends on the spectral parameter λ. Let us expand the determinant $\det \hat{\mathscr{A}}$ in powers of λ:

$$\det \hat{\mathscr{A}} = 1 - \lambda h \sum_{i=1}^{N} K_{ii} + \lambda^2 h^2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} K_{ii} K_{jj} - \lambda^2 h^2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} K_{ij} K_{ji} + \ldots$$



In the expansion above the matrix elements from which the factor λ is taken from are indicated by black dots. Here we use the expression for the determinant through permutations. We grouped the terms according to the cycle structure in the permutation, the cycles are indicated by thin lines. The vast majority of factors [in the case of fine discretization grid] in the corresponding to a permutation product of the matrix elements are coming from the diagonal.

In order to get a relatively comprehensible expression for $\det\hat{\mathcal{A}}$, it is convenient to group the λ^n terms differently. Let us say we have a λ^n term that contains $K_{i_1,j_1}, K_{i_2,j_2}, \dots, K_{i_n,j_n}$ as factors. All other factors in the corresponding to a permutation product of the matrix elements do not contain λ, so they come from the diagonal. Thus, the sets of i-indices and j-indices do coincide, and the mapping $\sigma : i_k \mapsto j_k$, $1 \le k \le n$, is a permutation of n objects.

In order to account for all λ^n terms, let us first choose the values $1 \le i_1 < i_2 < \dots < i_n \le N$ and harvest all the λ^n terms that use the discretized kernel values from these rows and columns. Then we sum over all possible values of $i_1, i_2, \dots, i_n$. We get

$$\det\hat{\mathcal{A}} = 1 + \sum_{n=1}^{\infty}(-\lambda h)^n \sum_{i_1=1}^{N-n+1} \sum_{i_2=i_1+1}^{N-n+2} \cdots \sum_{i_n=i_{n-1}+1}^{N} \begin{bmatrix} K_{i_1,i_1} & K_{i_1,i_2} & \cdots & K_{i_1,i_n} \\ K_{i_2,i_1} & K_{i_2,i_2} & \cdots & K_{i_2,i_n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{i_n,i_1} & K_{i_n,i_2} & \cdots & K_{i_n,i_n} \end{bmatrix}$$

By making the discretization grid finer and finer ($h\sum \to \int$), we finally obtain

$$\det\hat{\mathcal{A}} \xrightarrow{N\to\infty} \textit{Fredholm determinant } D_\lambda = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} A_n$$

$$A_0 := 1, \qquad A_n := \int_a^b \mathrm{d}y_1 \int_a^b \mathrm{d}y_2 \dots \int_a^b \mathrm{d}y_n \; K\begin{pmatrix} y_1, & y_2, & \dots, & y_n \\ y_1, & y_2, & \dots, & y_n \end{pmatrix}$$

$$K\begin{pmatrix} x_1, & x_2, & \dots, & x_n \\ y_1, & y_2, & \dots, & y_n \end{pmatrix} := \det \begin{bmatrix} K(x_1\,|\,y_1) & K(x_1\,|\,y_2) & \cdots & K(x_1\,|\,y_n) \\ K(x_2\,|\,y_1) & K(x_2\,|\,y_2) & \cdots & K(x_2\,|\,y_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_n\,|\,y_1) & K(x_n\,|\,y_2) & \cdots & K(x_n\,|\,y_n) \end{bmatrix}$$

Here we used the fact that there are $n!$ orderings of n objects:

$$\underbrace{\int_a^b \mathrm{d}y_1 \int_{y_1}^b \mathrm{d}y_2 \dots \int_{y_{n-1}}^b \mathrm{d}y_n}_{\text{so } y_1<y_2<\dots<y_n} \; \underbrace{F(y_1, y_2, \dots, y_n)}_{\text{a symmetric function}} = \frac{1}{n!} \int_a^b \mathrm{d}y_1 \int_a^b \mathrm{d}y_2 \dots \int_a^b \mathrm{d}y_n \; F$$

Let us now find out the behavior of $\det \hat{\mathscr{B}}_i$. There are two types of terms: the ones that contain f_i, and the ones that do not (they then have to contain some f_j with $j \neq i$). The f_i-type terms provide the contribution to $\det \hat{\mathscr{B}}_i$ which is f_i multiplied by the determinant of the matrix $\hat{\mathscr{B}}_i$ with i^{th} column (containing f's) and the i^{th} row removed. This determinant has the same structure as $\det \hat{\mathscr{A}}$, just one grid point is remived. In the limit $N \to \infty$ it tends to the Fredholm determinant D_λ.

To take into account all f_j-type terms, we need to sum over j. As the factor f_j comes from non-diagonal matrix element of $\hat{\mathscr{B}}_i$, we necessarily have at least one more non-diagonal factor which is going to contain h. This $h \sum$ will be converted to the integration over $[j \to] y$.

Let us say we have a λ^{n+1} term that contains $K_{i_1,j_1}, K_{i_2,j_2}, \ldots, K_{i_{n+1},j_{n+1}}$, and f_j (an element from the i^{th} column) as factors. As before, all other factors in the corresponding product of the matrix elements of $\hat{\mathscr{B}}_i$ come from the diagonal; and the sets of row and column numbers do coincide. As $i \neq j$, one of $i_1, i_2, \ldots, i_{n+1}$ is equal to i, and one of $j_1, j_2, \ldots, j_{n+1}$ is equal to j. We have to sum over all possible positions of the remaining n indices. We get

$$\left(\frac{f_j \text{ terms}}{\lambda} \ \text{ in } \ \det \hat{\mathscr{B}}_i\right) \xrightarrow{N\to\infty} \text{minor } D_\lambda(x\,|\,y) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} B_n(x\,|\,y)$$

$$B_n(x\,|\,y) := \int_a^b \mathrm{d}y_1 \int_a^b \mathrm{d}y_2 \ldots \int_a^b \mathrm{d}y_n \; K\begin{pmatrix} x, & y_1, & y_2, & \ldots, & y_n \\ y, & y_1, & y_2, & \ldots, & y_n \end{pmatrix}$$

also $B_0(x\,|\,y) := K(x\,|\,y)$. We write the solution as

$$u(x) = \underbrace{f(x)}_{f_i \text{ terms}} + \underbrace{\lambda \overbrace{\int_a^b \mathrm{d}y}^{\text{summation over } j} \; R_\lambda(x\,|\,y)\, f(y)}_{f_j \text{ terms}}$$

The function $R_\lambda(x\,|\,y) = D_\lambda(x\,|\,y)\,/D_\lambda$ is called the *Fredholm resolvent.*

The objects A_n and $B_n(x\,|\,y)$ satisfy the following recurrent relations:

$$A_0 = 1, \qquad B_0(x\,|\,y) = K(x\,|\,y), \qquad A_n = \int_a^b \mathrm{d}y\; B_{n-1}(y\,|\,y)$$

$$B_n(x\,|\,y) = K(x\,|\,y)\, A_n \; - \; n \int_a^b \mathrm{d}z\; K(x\,|\,z)\, B_{n-1}(z\,|\,y)$$