Consider the linear problem $u''(x) = f(x)$ with $u(0) = u(1) = u'(0) = u'(1) = 0$ boundary conditions. For which $f(x)$ there is a solution? Find the [modified] Green's function $K(x \mid y)$.

Let us calculate the adjoint $\hat{L}^\dagger$:

$$\langle v, \hat{L}u \rangle = \int_0^1 \mathrm{d}x\, v^*(x) u''(x) = \cancel{v^*(x) u'(x)\Big|_0^{\pi/2}}^{\,0} - \int_0^1 \mathrm{d}x\, \big(v'(x)\big)^* u'(x) =$$

$$= \cancel{-\big(v'(x)\big)^* u(x)\Big|_0^1}^{\,0} + \int_0^1 \mathrm{d}x\, \big(v''(x)\big)^* u(x)$$

thus $\hat{L}^\dagger = \mathrm{d}^2/\mathrm{d}x^2$ with no boundary conditions on $v(x)$. Arbitrary linear function is a zero mode of $\hat{L}^\dagger$, so there are two [linear independent] zero modes, *e.g.*, $v_1(x) = 1$ and $v_2(x) = x$. In order for solution to exist, the r.h.s. $f(x)$ should be orthogonal to both of them.

The equation $\hat{L}K(x \mid y) = \delta(x - y)$ should be modified — the r.h.s. should be orthogonalized to the zero modes of $\hat{L}^\dagger$. The space of zero modes is two-dimensional, and there is a plenty of ways to choose a basis there. Let us demonstrate that two different choices lead to the same modification.

Let us take the pair $v_1(x) = 1$ and $v_2(x) = x$, and then orthogonalize it by the Gram–Schmidt process. We end up with the orthonormalized functions as $U_{0,1}(x) = 1$ and $U_{0,2}(x) = \sqrt{12}(x - 1/2)$.

Now let us take the same pair but in different order: $v_2(x)$ goes first. We end up with the orthonormalized functions as $U_{0,3}(x) = \sqrt{3}x$ and $U_{0,4}(x) = 3x - 2$.

The modification of the r.h.s. goes as

$$\begin{aligned}
\delta(x-y) \longrightarrow \delta(x-y) - U_{0,1}(x)U^*_{0,1}(y) - U_{0,2}(x)U^*_{0,2}(y) &= \\
= \delta(x-y) - 1 - 12(x-1/2)(y-1/2) &= \\
= \delta(x-y) - (12xy - 6x - 6y + 4) &= \\
= \delta(x-y) - 3xy - (3x-2)(3y-2) &= \\
= \delta(x-y) - U_{0,3}(x)U^*_{0,3}(y) - U_{0,4}(x)U^*_{0,4}(y) &\longleftarrow \delta(x-y)
\end{aligned}$$

Now let is find the modified Green's function $K(x \mid y)$. We have

$$K(x \mid y) = -2x^3y + x^3 + 3x^2y - 2x^2 + \begin{cases} Ax + B, & x < y; \\ Cx + D, & x > y \end{cases}$$

where the 4 constants A, B, C, D are found from the 4 boundary conditions. We find $A = B = 0$ from $K(0 \mid y) = (\mathrm{d}/\mathrm{d}x)K(x \mid y)\big|_{x=0} = 0$. Also $y - 1 + C + D = 0$ from $K(1 \mid y) = 0$, and $C = 1$ from $(\mathrm{d}/\mathrm{d}x)K(x \mid y)\big|_{x=1} = 0$. (Thus, $D = -y$.)

So far we did not use the conditions that the Green's function is continuous and has the derivative at $x = y$ to jump by 1. Because we so carefully did prepare the r.h.s., this should automatically be satisfied. As $C = 1$ we have the correct jump, and as $Cx + D = x - y$ is equal to 0 at $x = y$ the Green's function is indeed continuous. Our final expression for the Green's function is

$$K(x \mid y) = -2x^3 y + x^3 + 3x^2 y - 2x^2 + H(x - y) \cdot (x - y)$$

Let us now interpret the obtained Green's function as the integral kernel of the pseudoinverse of $\hat{L}$. For that let us find the "singular value decomposition" of $\hat{L}$. The "right singular eigenvectors" are the eigenfunctions of $\hat{L}^\dagger \hat{L} = \mathrm{d}^4/\mathrm{d}x^4$, a self-adjoint and positive definite operator acting in "small" space of functions satisfying $u(0) = u(1) = u'(0) = u'(1) = 0$ boundary conditions:

$$\langle v, \hat{L}u \rangle = \int_0^1 \mathrm{d}x\; v^*(x)\, u''''(x) = (\text{non-integral terms}) + \int_0^1 \mathrm{d}x\; \left(v''''(x)\right)^* u(x)$$

In non-integral terms we have 3 differentiations somehow distributed between u and v. In any term one of the functions is differentiated just once or not differentiated at all, rendering this term being equal to 0.

Consider $k^4 > 0$ to be an eigenvalue of $\hat{L}^\dagger \hat{L}$. Then the corresponding eigenfunction $V(x)$ is a linear combination of $\cosh kx$, $\sinh kx$, $\cos kx$, and $\sin kx$. From the boundary conditions $V(0) = V'(0) = 0$ it should look like

$$V(x) = A(\cosh kx - \cos kx) + B(\sinh kx - \sin kx)$$

where A and B are some constants. In order to satisfy another pair of boundary conditions, $V(1) = V'(1) = 0$, the constants A and B should solve a homogeneous system of two linear equations

$$\begin{bmatrix} V(1) \\ V'(1) \end{bmatrix} = \begin{bmatrix} \cosh k - \cos k & \sinh k - \sin k \\ \sinh k + \sin k & \cosh k - \cos k \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which has a non-zero solution only if $\cosh k \cos k = 1$. There are infinitely many suitable values of k, which we will denote as $k_1[\approx 4.73] < k_2[\approx 7.85] < \ldots$ We have $\lim_{i\to\infty} k_i = +\infty$ (the operator $\hat{L}$ is unbounded), with $k_i \approx \pi(i{+}1/2)$ for [even not so very] large i.

The normalized eigenfunction $V_i(x)$ with an eigenvalue k_i looks like

$$V_i(x) = (\cosh k_i x - \cos k_i x) - \frac{\cosh k_i - \cos k_i}{\sinh k_i - \sin k_i}(\sinh k_i x - \sin k_i x)$$

The left "singular eigenfunctions" can be obtained as $U_i = \hat{L} V_i / k_i^2 = V_i'' / k_i^2$, with the result

$$U_i(x) = (\cosh k_i x + \cos k_i x) - \frac{\cosh k_i - \cos k_i}{\sinh k_i - \sin k_i}(\sinh k_i x + \sin k_i x)$$

The two left singular eigenfunctions that correspond to the zero singular value are $U_{0,1}(x) = 1$ and $U_{0,2}(x) = \sqrt{3}(2x-1)$. Let us check that they are orthogonal to all $U_i(x)$:

$$\langle U_{0,j}, U_i \rangle = \frac{1}{k_i^2} \int_0^1 \mathrm{d}x \, U_{0,j}^*(x) \, V_i''(x) = \int_0^1 \mathrm{d}x \, \left(U_{0,j}''(x)\right)^* V_i(x) = 0$$

Non-integral terms in integration by parts are zero here as $V_i(0) = V_i(1) = V_i'(0) = V_i'(1) = 0$.

We should have

$$L(x \,|\, y) = \sum_{i=1}^{\infty} U_i(x) \, k_i^2 \, V_i^*(y) \quad \text{and} \quad K(x \,|\, y) = \sum_{i=1}^{\infty} \frac{V_i(x) \, U_i^*(y)}{k_i^2}$$

In "matrix" form the operator $\hat{L}$ could be written as

$$L(x \,|\, y) = \begin{bmatrix} \cdots & U_3(x) & U_2(x) & U_1(x) & U_{0,2}(x) & U_{0,1}(x) \end{bmatrix} \begin{bmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & k_3^2 & 0 & 0 \\ \cdots & 0 & k_2^2 & 0 \\ \cdots & 0 & 0 & k_1^2 \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vdots \\ V_3^*(y) \\ V_2^*(y) \\ V_1^*(y) \end{bmatrix}$$

while the Green's function $\hat{K}$ is obtained by dropping the zero singular values, *i.e.*, removing the last two rows of $\hat{\Sigma}$ and the last two columns of $\hat{U}$ (zero modes of $\hat{L}^\dagger$), and inverting the resulting matrix.

All this is true if the basus of U-functions is complete, *i.e.*, we have

$$\delta(x-y) = U_{0,1}(x) U_{0,1}^*(y) + U_{0,2}(x) U_{0,2}^*(y) + \sum_{i=1}^{\infty} U_i(x) \, U_i^*(y)$$

as the decomposition of identity operator.