Homework 2, problem 4

Evaluate $\displaystyle\int_0^\infty \mathrm{d}x\, \frac{\ln^2 x}{(x+1)(x+2)}$.

The keyhole contour Γ is to be used, with the function 1) $f(z) = \log^3 z/(z+1)(z+2)$, and 2) $f(z) = z^\alpha/(z+1)(z+2)$. In both ways let us think $0 < \arg z < 2\pi$. Let us denote

$$J(\alpha) = \int_0^\infty \mathrm{d}x\, \frac{x^\alpha}{(x+1)(x+2)}, \qquad J_n = \int_0^\infty \mathrm{d}x\, \frac{\ln^n x}{(x+1)(x+2)} = \left.\frac{\mathrm{d}^n J(\alpha)}{\mathrm{d}\alpha^n}\right|_{\alpha=0}$$

In way 1) we have

$$2\pi\mathrm{i}\underbrace{\left(\frac{(\mathrm{i}\pi)^3}{-1+2} + \frac{(\ln 2 + \mathrm{i}\pi)^3}{-2+1}\right)}_{\text{residues}} = \int_\Gamma \mathrm{d}z\, f(z) = \int_0^\infty \mathrm{d}x\, \frac{\overbrace{\ln^3 x}^{\text{above}} - \overbrace{(\ln x + 2\pi\mathrm{i})^3}^{\text{below the cut}}}{(x+1)(x+2)} = -6\pi\mathrm{i}J_2 + 12\pi^2 J_1 + 8\pi^3\mathrm{i}J_0$$

Noting that $J_0 = \ln 2$ (*e.g.*, by partial fractions), we have

$$2\pi\mathrm{i}\Big(-\ln^3 2 - \underbrace{3\pi\mathrm{i}\ln^2 2}_{\text{gives } J_1} + 3\pi^2 \ln 2\Big) = -6\pi\mathrm{i}J_2 + 12\pi^2 J_1 + 8\pi^3\mathrm{i}\ln 2$$

from where $J_2 = \boxed{\frac{1}{3}(\ln^2 2 + \pi^2)\ln 2}$.

In way 2) we have

$$2\pi\mathrm{i}\underbrace{\left(\frac{\mathrm{e}^{\pi\mathrm{i}\alpha}}{-1+2} + \frac{2^\alpha \mathrm{e}^{\pi\mathrm{i}\alpha}}{-2+1}\right)}_{\text{residues}} = \int_\Gamma \mathrm{d}z\, f(z) = \big(\underbrace{1 - \mathrm{e}^{2\pi\mathrm{i}\alpha}}_{\text{above and below the cut}}\big)J(\alpha)$$

from where

$$J(\alpha) = \frac{\pi\left(2^\alpha - 1\right)}{\sin \pi\alpha} \approx \frac{1 + \alpha\ln 2 + \frac{\alpha^2\ln^2 2}{2} + \frac{\alpha^3 \ln^3 2}{6} - 1}{\alpha\left(1 - \frac{\pi^2\alpha^2}{6}\right)} \approx \underbrace{\ln 2}_{J_0} + \underbrace{\frac{\alpha\ln^2 2}{2}}_{\text{gives } J_1} + \frac{\alpha^2\ln^3 2}{6} + \frac{\pi^2\alpha^2}{6}\ln 2$$

from where $J_2 = \boxed{\frac{1}{3}(\ln^2 2 + \pi^2)\ln 2}$.