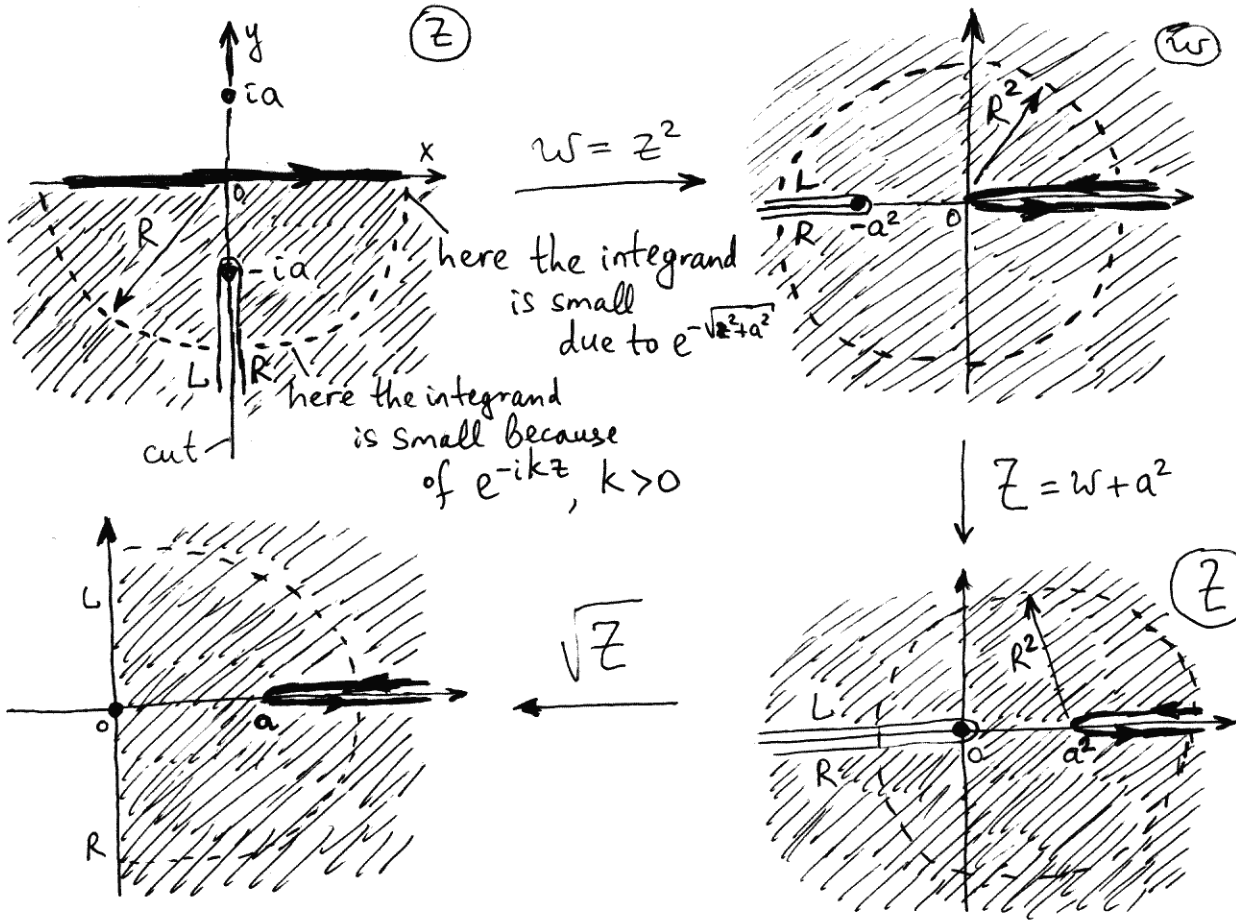
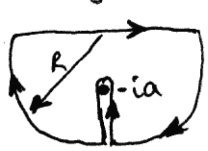
(a) $\hat{f}(k) = \int_{-\infty}^{\infty} dx\; e^{-ikx-\sqrt{x^2+a^2}} = \left\{ \begin{array}{l} \hat{f}(k) \text{ is real, even} \\ \text{assume } k>0 \end{array} \right\} \overset{(*)}{=}$



$$\int dz\, e^{-ikz-\sqrt{z^2+a^2}} \underset{\substack{\text{no singularities} \\ \text{inside the contour}}}{=} 0$$



the integral over the circular part goes to 0 when $R \to \infty$

$$\overset{(*)}{=} \int dz\, e^{-ik\underset{x+iy}{\underset{\shortparallel}{z}}-\sqrt{z^2+a^2}} = \int_{-\infty}^{-a} dy\; \overset{e^{-ikz}}{\overset{\shortparallel}{e^{ky}}} \Big(i \overset{e^{-\sqrt{z^2+a^2}}}{\overset{\shortparallel}{\underset{L}{e^{-i\sqrt{y^2-a^2}}}}} - i \underset{R}{e^{i\sqrt{y^2-a^2}}} \Big)$$

$dz = i\,dy$

$$\underset{y\to -y}{\hat{f}(k) =} 2\int_a^\infty dy\, \underset{\hat{f}(k)\text{ is even}}{e^{-|k|y}} \sin\left(\sqrt{y^2-a^2}\right) = \left\{ \xi = y-a \right\} =$$

$$= 2\,\underset{\text{decay with } \gamma = \boxed{a}}{e^{-|k|a}} \underbrace{\int_0^\infty d\xi\, e^{-|k|\xi} \sin\sqrt{\xi(\xi+2a)}}$$

For large k we effectively have $\xi \lesssim \frac{1}{|k|}$ (otherwise $e^{-|k|\xi}$ is too small), and

$\underbrace{\int\ldots} \approx \int_0^\infty d\xi\, e^{-|k|\xi}\sqrt{2a\xi} = \sqrt{\frac{\pi a}{2|k|^3}}$ — depends on k algebraically (i.e., slowly), not exponentially. Note that $\underbrace{\int\ldots}$ decays as $1/|k|^{3/2}$ — this is due to square root singularities of $f(x)$ at $x = \pm ia$ — similar to what happens in Problem 4.

(b) The closest singularity to the real axis (which is $x = ia$ or $x = -ia$) is away from it by distance a, so $\gamma = \boxed{a}$.

In $\hat{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x)$ we can lift or sink the contour (which is the real axis) by a, which results in making e^{-ikx} by $e^{-|k|a}$ times smaller.