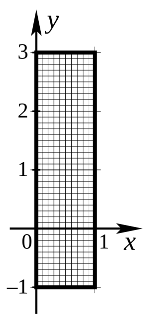
Exponential function

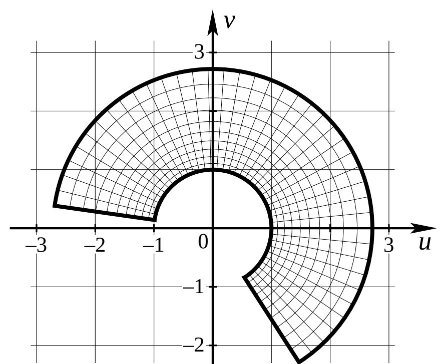
Let us define an exponential function $\exp(z)$ as

$$\exp(z) := \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n.$$

This definition coincides with the standard one on real numbers.

$$\begin{aligned}
\exp\left(x+\mathrm{i}\,y\right) &= \lim_{n\to\infty}\left(1+\frac{x+\mathrm{i}\,y}{n}\right)^n = \\
&\quad \left|\ \text{if } n>|x|\text{, then } -\pi/2 < \arg\left(1+(x+\mathrm{i}\,y)/n\right) < \pi/2\ \right| \\
&= \lim_{n\to\infty}\left(\left(\left(1+\frac{x}{n}\right)^2+\left(\frac{y}{n}\right)^2\right)^{n/2}\left(\left(\cos+\mathrm{i}\,\sin\right)\left(\arctan\frac{y/n}{1+x/n}\right)\right)\right)^n = \\
&= \lim_{n\to\infty}\left(1+\frac{2x}{n}+\frac{x^2+y^2}{n^2}\right)^{n/2}\cdot\left(\cos+\mathrm{i}\,\sin\right)\left(n\arctan\frac{y/n}{1+x/n}\right) = \\
&= \exp(x)\cdot\left(\cos y+\mathrm{i}\,\sin y\right).
\end{aligned}$$





The multiplicative property $\exp(z_1+z_2) = \exp(z_1)\cdot\exp(z_2)$ for any complex numbers z_1, z_2 is still valid. This can be checked directly, as

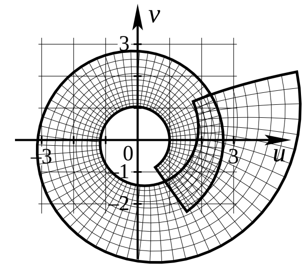
$$\left(\cos y_1+\mathrm{i}\,\sin y_1\right)\cdot\left(\cos y_2+\mathrm{i}\,\sin y_2\right) = \cos(y_1+y_2)+\mathrm{i}\,\sin(y_1+y_2).$$

Another way to prove this is

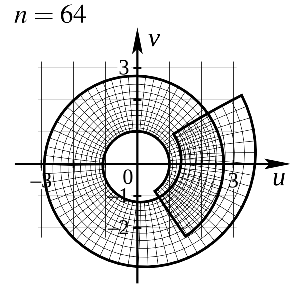
$$\begin{aligned}
\exp(z_1+z_2) &= \lim_{n\to\infty}\left(1+\frac{z_1+z_2}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{z_1+z_2}{n}+\frac{z_1 z_2}{n^2}\right)^n = \\
&= \lim_{n\to\infty}\left(1+\frac{z_1}{n}\right)^n\left(1+\frac{z_2}{n}\right)^n = \exp(z_1)\cdot\exp(z_2).
\end{aligned}$$

At the right the images of the rectangle $0\le x\le 1$, $-1\le y\le 7$ are shown for the mappings $w=(1+z/n)^n$ with different n.

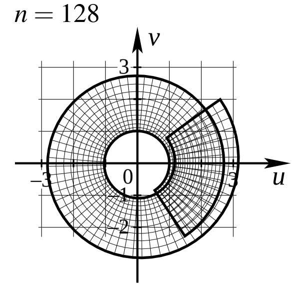
$n=32$



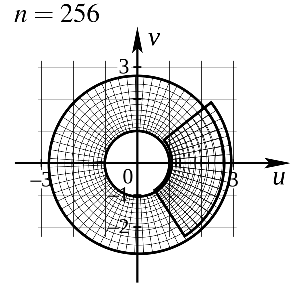
$n=64$



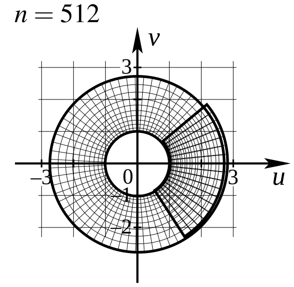
$n=128$



$n=256$



$n=512$



Another expression for exponential function $\exp(z)$ is

$$\exp(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \ldots + \frac{z^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The multiplicative property $\exp(z_1+z_2) = \exp(z_1)\cdot\exp(z_2)$ is easily checked:

$$\exp(z_1+z_2) = \sum_{n=0}^{\infty} \frac{(z_1+z_2)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z_1^k z_2^{n-k} =$$

$$= \Big| m = n-k \Big| = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{z_1^k z_2^m}{k!m!} = \exp(z_1)\cdot\exp(z_2).$$

We have $\exp\big(x+\mathrm{i}y\big) = \exp(x)\exp(\mathrm{i}y)$, and

$$\exp(\mathrm{i}y) = \sum_{n=0}^{\infty} \frac{(\mathrm{i}y)^n}{n!} = \sum_{m=0}^{\infty} \frac{\mathrm{i}^{2m} y^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{\mathrm{i}^{2m+1} y^{2m+1}}{(2m+1)!} =$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m y^{2m}}{(2m)!} + \mathrm{i} \sum_{m=0}^{\infty} \frac{(-1)^m y^{2m+1}}{(2m+1)!} = \cos y + \mathrm{i}\sin y.$$

At the right the images of the rectangle $0 \le x \le 1$, $-1 \le y \le 3$ are shown for the mappings

$$w = \sum_{n=0}^{N} \frac{z^n}{n!}$$

with different N.