Why do we need complex numbers?

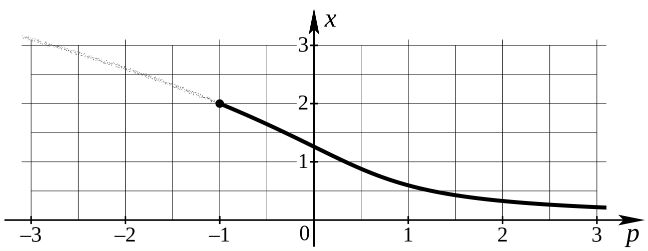
Cubic equation

Consider we are to solve a cubic equation $ax^3+bx^2+cx+d=0$. We assume that $a\neq 0$ (otherwise the equation is quadratic at best), so we can divide this equation by a: $x^3+Bx^2+Cx+D=0$, where $\{B,C,D\}=\{b,c,d\}/a$. Then let us make a variable change $X=x+B/3$. That will kill quadratic term and produce a form $X^3+PX+Q=0$ — *depressed* cubic (with $P=C-B^2/3$, $Q=D-BC/3+2B^3/27$). Now if $Q=0$, then $X=0$ is a solution, and the rest is quadratic equation $X^2+P=0$. If $Q\neq 0$, then let us rescale the variable: $X=-(Q/2)^{1/3}x$, and the equation gets the following form $x^3+\left[P/(Q/2)^{2/3}\right]x-2=0$. The only remained parameter of the equation is

$$p=\frac{P}{(Q/2)^{2/3}}=\left(3ac-b^2\right)\Big/\left(b^3-\frac{9abc}{2}+\frac{27a^2d}{2}\right)^{2/3}$$

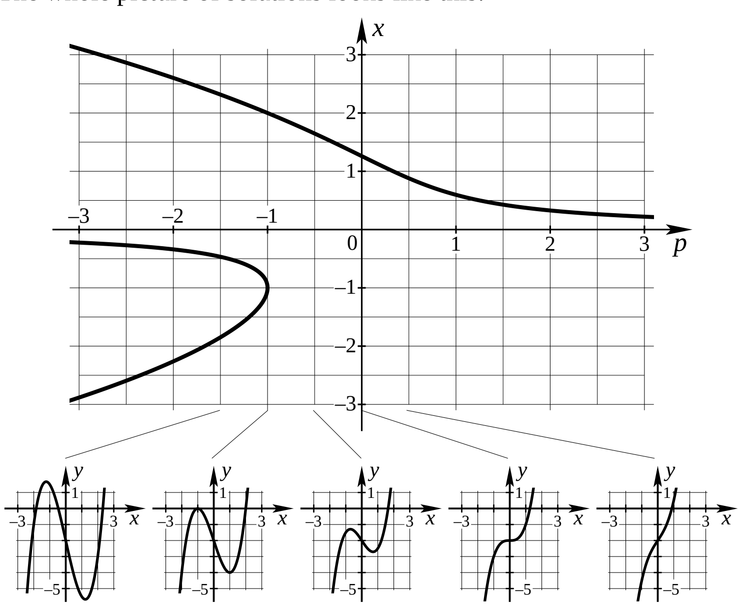
Now we are dealing with the equation $x^3+3px-2=0$. It can be checked directly that

$$x=\left(1+\sqrt{1+p^3}\right)^{1/3}+\left(1-\sqrt{1+p^3}\right)^{1/3}$$



gives us a root. The expression is defined only if $p\geq -1$, otherwise the argument of the square root $1+p^3$ is negative. The curve $x(p)$ describing a root nevertheless happily continues to the left not paying attention to the fact that $1+p^3$ became negative. This part of the curve on the plot is “cloudy” on the graph. We can draw the curve there as follows: Take any $x>2$ and calculate $p=(2-x^3)/3x$ that will be less than -1. That x will give us a solution for this p.

The whole picture of solutions looks like this:



At $p=-1$ in addition to $x=2$ the 2nd root appears: $x=-1$, while there are three roots if $p<-1$.