# Quadratic Forms and Automorphic Forms

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# Part I

# Notes

# Chapter 1

# Background on Quadratic Forms

## 1.1 Definitions of Quadratic Forms

In this chapter we give some basic definitions and ideas used to understand quadratic forms and the numbers they represent. We define a **quadratic form** $Q(\vec{x})$ **over a ring** $R$ to be a degree 2 homogeneous polynomial

$$Q(\vec{x}) = \sum_{1 \leq i \leq j \leq n} c_{ij} x_i x_j$$

in $n$ variables with coefficients $c_{ij}$ in $R$.

A quadratic form can also be viewed as an even symmetric matrix $A = (a_{ij})$ where $a_{ij} = c_{ij} + c_{ji}$ (under the convention that $c_{ij} = 0$ if $i > j$). Since this is the Hessian matrix of second order partial derivatives of $Q(\vec{x})$, we refer to $A$ as the **Hessian matrix** of $Q$.

Sometimes it is useful to consider $Q(\vec{x})$ as coming from a bilinear form $B(\vec{x}, \vec{y})$ by the formula $Q(\vec{x}) = B(\vec{x}, \vec{x})$, and we refer to $B = (b_{ij})$ as the **bilinear matrix** of $Q$. From the polarization identity

$$B(\vec{x} + \vec{y}, \vec{x} + \vec{y}) = B(\vec{x}, \vec{x}) + 2B(\vec{x}, \vec{y}) + B(\vec{y}, \vec{y})$$

we can see that the bilinear matrix is exactly half of the Hessian matrix (i.e. $B = \frac{1}{2}A$). While it is conceptually easier to think about $Q$ as coming from a bilinear form, this perspective requires division by 2. So while the Hessian matrix always has coefficients in $R$, *the bilinear matrix may not in general have coefficients in* $R$ *unless* $2 \in R^\times$. For this reason, it is often preferable to use the Hessian formulation when working over rings, and in particular when $R = \mathbb{Z}$.

Another perspective on quadratic forms is to think of them as **free quadratic** $R$**-modules**, by which we mean an $R$-module $M \cong R^n$ equipped with a quadratic function

$$\phi : M \to R$$

satisfying $\phi(a\vec{x}) = a^2\phi(\vec{x})$ and for which $\phi(\vec{x}+\vec{y}) - \phi(\vec{x}) - \phi(\vec{y})$ is a bilinear form. This perspective allows us to think of $Q$ as a quadratic lattice, which we may then choose to embed in a larger quadratic space in various ways.

Given a lattice $L$ in a vector space $V$ equipped with a non-degenerate symmetric bilinear form $B(\vec{x},\vec{y})$, we define the **dual lattice** $L^{\#}$of $L$ to be the set of vectors of $V$ that pair integrally with all $\vec{w} \in$ , i.e.

$$L^{\#} := \{\vec{w} \in V \mid B(\vec{v},\vec{w}) \in \mathbb{Z} \text{ for all } v \in L\}.$$

If $B(\vec{v},\vec{v})$ is integer-valued for all $\vec{v} \in L$ then we have $L \subseteq L^{\#}$. For such lattices we know that the matrix of $B$ in any basis of $L$ is symmetric with integer coefficients and even diagonal, and we call such matrices **even matrices**. We then define the **level** of $L$ to be the smallest $N \in \mathbb{N}$ so that the matrix $NA^{-1}$ is also even. The level is a very useful invariant of $L$ which appears when we take dual lattices (because $A^{-1}$ is the matrix of basis vectors for the dual basis of the given basis $\mathcal{B}$ of $L$ in the coordinates of $\mathcal{B}$), and in particular it will figure prominently in the theory of theta functions.

## 1.2 Equivalence of Quadratic Forms

We say that two quadratic forms are **equivalent over** $R$ if there is an invertible linear change of variables with coefficients in $R$ from one to the other preserving their values.

We represent this as left multiplication by a matrix $M$ with respect to the generators $e_i$ of the free module $R^n$ – i.e.

$$L(\vec{x}) = M\vec{x}.$$

In terms of the matrices $A$ and $B$, this gives the equivalence

$$A \mapsto M^tAM$$

where

$$Q(\vec{x}) = \tfrac{1}{2}\vec{x}^{\,t}A\vec{x} = \vec{x}^tB\vec{x}$$

We say that two quadratic modules are isomorphic is there is an isomorphism between the modules commuting with the quadratic functions.

## 1.3 Direct Sums and Scaling

Two important constructions for making new quadratic forms from known ones are the operations of scaling and direct sum. Given $a \in R$ and a quadratic form $Q(\vec{x})$ defined over $R$, we can define a new **scaled** quadratic form $a \cot Q(\vec{x})$ which is also defined over $R$. We can detect if a quadratic form is scaled by looking at its values generated either as a bilinear form, or as a quadratic form. We therefore define the **scale** and **norm** of $Q$ by

$$\mathrm{Scale}(Q) := \{B(\vec{x},\vec{y}) \mid \vec{x},\vec{y} \in R^n\} \tag{1.1}$$

$$\mathrm{Norm}(Q) := \{Q(\vec{x}) \mid \vec{x} \in R^n\} \tag{1.2}$$

Another interesting construction of new quadratic forms is the direct sum of two given quadratic forms. Given $Q_1(\vec{x}_1)$ and $Q_2(\vec{x}_2)$ in $n_1$ and $n_2$ distinct variables over $R$ respectively, we define their **direct sum** $Q_1 \oplus Q_2$ in $n_1 + n_2$ variables as

$$(Q_1 \oplus Q_2)(\vec{x}) := Q_1(\vec{x}_1) + Q_2(\vec{x}_2)$$

where $\vec{x} := (\vec{x}_1, \vec{x}_2)$.

## 1.4 Quadratic spaces

**Make a comment about the characteristic of $K$ not being 2 throughout!**

Quadratic forms become much simpler to study when the base ring $R$ is actually a field $K$, in which case we know that all finite-dimensional $R$-modules are free (i.e. every finite-dimensional vector space has a basis), and there is exactly one of each dimension $n$. This allows us to replace commutative algebra with linear algebra when studying quadratic forms. For this reason we define a **quadratic space** to be a quadratic form $Q$ over a field $K$.

As an example of this phenomenon, we state Witt's Theorem, which classifies isometric quadratic subspaces within a given quadratic space.

**Theorem 1.4.1** (Witt's Theorem)**.** *Suppose that $U$ and $U'$ are isometric (quadratic) subspaces of a quadratic space $V$. Then any isometry $\alpha : U \to U'$ extends to an isometry $\alpha : V \to V$.*

*Proof.* This is proved in almost every quadratic forms book, e.g. Cassels's book [3, Thrm 4.1 on p21], Shimura's book, and Lam's book. □

Another useful theorem for a quadratic space $(\mathrm{char}(K) \neq 2)$, we can always find an orthogonal basis, which puts the matrices associated to it in diagonal form.

**Theorem 1.4.2** (Orthogonal splitting/diagonalization)**.** *Any quadratic space $V$ admits an orthogonal basis.*

*Proof.* This is proved in Cassels's book [3, Lemma 1.4 on p13], where he refers to this as a "normal basis". □

For a quadratic space $Q$, the determinant $\det(Q)$ is an element of the square-classes of $K$. We say that $Q$ is **degenerate** if $\det(Q) = 0$, otherwise we say that $Q$ is **non-degenerate**. The next theorem states that we can always reduce a degenerate quadratic space to a non-degenerate space by splitting off a trivial/zero form. We define the **radical of** $Q$ to be the maximal subspace of $Q$ perpendicular $(\mathrm{char}(K) \neq 2)$ to the entire space.

**Lemma 1.4.3** (Radical Splitting)**.** *Every quadratic space can be written as a direct sum of a zero space and a regular space.*

*Proof.* This is given in Cassels's book [3, Lemma 6.1 on p28]. □

A slightly weaker kind of splitting comes from looking at quadratic subspaces of $V$ on which the quadratic form restricts to the zero form. These are called **isotropic subspaces** of $Q$, and for any non-degenerate form they can be found in complementary pairs. We also say that any non-zero vector $\vec{v}$ in an isotropic subspace is **isotropic** (i.e. $Q(\vec{v}) = 0, \vec{v} \neq \vec{0}$), otherwise we say that $\vec{v}$ is **anisotropic**.

**Theorem 1.4.4** (Isotropic Splitting)**.** *Suppose $Q$ is a non-degenerate quadratic space. Then for every isotropic subspace $Q_1$ of $Q$ we can find a complementary subspace $Q_1'$ of $Q$ so that $Q = Q_1 \oplus Q_1' \oplus Q_2$ where $Q_1 \oplus Q_1'$ is the hyperbolic plane $H_{2n_1}$ (i.e. $Q_1 \cong Q_2 \cong 0$ and we can find a basis $\{e_i, f_i\}$ for them satisfying $B(e_i, f_j) = \delta_{ij}$).*

This is particularly useful when we take $Q_1$ to be an isotropic subspace of maximal dimension in $Q$, in which case we see that $Q_2$ has no (non-zero) isotropic vectors and therefore $Q_2$ is anisotropic.

## 1.5 Quadratic Forms over Local Fields

We now suppose that $Q$ is a quadratic space over one of the local fields $\mathbb{R}, \mathbb{C}$ or $\mathbb{Q}_p$ where $p$ is a prime number in $\mathbb{N}$. In this case we can attempt to classify quadratic spaces and rings in terms of certain invariants associated to them. The major result along these lines is that in addition to the dimension and determinant, at most one additional invariant is needed to classify quadratic spaces.

**Theorem 1.5.1.** *There is exactly one quadratic space over $\mathbb{C}$ of each dimension $n$.*

**Theorem 1.5.2.** *Quadratic spaces over $\mathbb{R}$ are in 1-1 correspondence with the pairs $(n, p_1)$ where $0 \leq p_1 \leq n$.*

*Proof.* This follows from the existence of an orthogonal basis, and that there is only one squareclass in $\mathbb{C}^\times$. Since $\mathbb{R}^\times$ has two squarclasses $\pm(\mathbb{R}^\times)^2$, we see that the diagonal elements can be chosen to be either 1 or -1. Since the number of hyperbolic planes is an invariant, this characterizes the number of $(1, -1)$ pairs on the diagonal, and then the remaining diagonal entries all have the same sign. Its orthogonal complement is anisotropic, and is either $1_{n-2r}$ or $-1_{n-2r}$ depending on the sign of the values it represents. □

The determinant can easily be computed from $p_1$, as well as the more standard (and equivalent) invariant replacing $p$ called the **signature of** $Q$, defined to be $p_1 - p_2$. However we chose to use $p_1$ here because the signature seems more complicated to think about and takes somewhat less natural values than $p_1$.

**Theorem 1.5.3.** *Quadratic space over* $\mathbb{Q}_p$ *are in 1-1 correspondence with the triples* $(n, d, c)$ *where* $n = \dim(Q)$, $d = \det(Q)$, *and* $c \in \{\pm 1\}$ *is the Hasse invariant of* $Q$*, under the restrictions that* $c = 1$ *if* $n = 1$ *or* $(n, d) = (2, -1)$.

*Proof.* This is proved in Cassels's book [3, Thrm 1.1 on p55]. □

Another area in which we can extract more information about quadratic forms over local fields is in terms of the **maximal anisotropic dimension of** $K$, which is defined to be the largest dimension of an anisotropic subspace of any quadratic space over $K$. This is sometimes called the **$u$-invariant** of $K$.

**Theorem 1.5.4.** *The maximal anisotropic dimensions of* $\mathbb{C}, \mathbb{R}$ *and* $\mathbb{Q}_p$ *are* $1, \infty$, *and* $4$.

This result can be understood in terms of the possible values that can be represented by a quadratic form of dimension $n$. Over $\mathbb{C}$ and $\mathbb{R}$ this follows from the classification theorem above.

This result for $\mathbb{R}$ says that it is possible for quadratic forms of any dimension to be anisotropic, which means that either $Q$ represents only positive or only negative values, and in these cases we say $Q$ is **positive definite** or **negative definite** respectively. When $Q$ is isotropic over $\mathbb{R}$ then it is either degenerate or represents both positive and negative values, in which case we say that $Q$ is **indefinite**.

For $\mathbb{Q}_p$ there is no notion of positive and negative, but one can concretely understand the $u$-invariant $u(\mathbb{Q}_p) = 4$ from the existence of certain (non-split) quaternion algebras at every prime p, which will be discussed in more detail in Lecture 3. For these quaternion algebras, their norm forms assume all values of $K$ and do not represent zero non-trivially.

## 1.6 Quadratic Forms over Local ($p$-adic) Rings of Integers

If we consider quadratic forms over the ring of integers $\mathbb{Z}_p$ of a $p$-adic field, then the classification theorem is more involved because the valuation and units will both play a role. The main result along these lines involves the notion of a Jordan splitting, which breaks $Q$ into a sum of pieces scaled by powers of $p$ which are as simple as possible.

**Theorem 1.6.1** (Jordan Decomposition)**.** *A quadratic form over* $\mathbb{Z}_p$ *can be written as a direct sum*

$$Q(\vec{x}) = \sum_{j \in \mathbb{Z}} p^j Q_j(\vec{x}_j)$$

*where* $Q_j(\vec{x}_j)$ *is a unimodular form (i.e. self-dual). If* $p > 2$ *then* $Q_j = u_i x^2$ *for some* $p$*-adic units* $u_j$, *and if* $p = 2$ *then each* $Q_j(\vec{x}_j)$ *is at most 2-dimensional.*

*Proof.* When $p > 2$ this is given in Cassels's book [3, Lemma 3.4 on p115], while $p = 2$ is given in a very explicit form in [3, Lemma 4.1 on p117]. It is given in a more canonical form in O'Meara's book [**?**] (for forms over the ring of integers of a number field), but in that context the exact summands at primes over $p = 2$ cannot be enumerated easily. $\square$

As a convention, we consider the ring of integers of $\mathbb{R}$ and $\mathbb{C}$ to be itself so there is nothing new to say in that situation.

## 1.7 Local-Global Results for Quadratic forms

A useful idea for studying quadratic forms (either over a number field $K$ or its ring of integers $\mathcal{O}_K$) is to consider them locally over all completions, and then try to use this local information to answer questions about the original form. While it is easy to pass from $Q$ to the form over its local completion $Q_v$ at the valuation $v$, it is more difficult to reverse this process to glue together a set of local $Q_v$ for all $v$ to obtain some global form $Q$. Along these lines, we have the following results which guarantee the existence of a global form under certain circumstances.

**Theorem 1.7.1** (Hasse-Minkowski Theorem)**.** *Given two quadratic forms $Q$ and $Q'$ defined over a $Q$, we have*

$$Q_1 \sim_{\mathbb{Q}} Q_2 \iff Q_1 \sim_{\mathbb{Q}_v} Q_2 \text{ for all places } v \text{ of } \mathbb{Q}$$

*Proof.* This is stated in Cassels's book [3, Thrm 1.3 on p77], but proved in [3, §6.7, p85–86]. $\square$

For quadratic forms under $\mathbb{Z}$-equivalence $\sim_{\mathbb{Z}}$ there is a similar local-global correspondence with local equivalence $\sim_{\mathbb{Z}_v}$ for all places $v$, but now the passage $Q \mapsto \{Q_v\}_v$ may not be 1-to-1 on equivalence classes. The multiplicity of the localization map $Q \mapsto \{Q_v\}_v$ under $\mathbb{Z}$-equivalence is called the **class number** $h_Q$ of the quadratic form $Q$, and the set of all forms with the same localization as $Q$ is called the **genus of** $Q$, so $h_Q = |\mathrm{Gen}(Q)|$.

It is a result of Siegel from the reduction theory of (either definite or indefinite) quadratic forms over $\mathbb{R}$ that $h_Q < \infty$. The class number of an indefinite quadratic form of dimension $n \geq 3$ is particularly simple to compute, and can be found in terms of a few local computations, but the class number of a definite form is considerably more complicated to understand exactly.

**Theorem 1.7.2.** *The class number $h_Q$ is finite.*

*Proof.* This follows from the reduction theory of quadratic forms, which shows that every class of quadratic forms over $\mathbb{Z}$ has some representative (of the same determinant) whose coefficients lie in a compact set. This together with the discreteness of the (integer) coefficients of $Q$ gives that there are only finitely many classes of quadratic forms of bounded discriminant. A proof can be found in [3, Thrm 1.1, p128 and Lemma 3.1, p135] $\square$

Interestingly, while indefinite forms appear more complicated on the surface, their arithmetic is usually *easier* to understand than that of definite forms, as can be the following theorems. The main idea is due to Eichler who discovered that the arithmetic of a certain simply connected algebraic group called the *spin group*, which is a double covering $\mathrm{SO}(Q)$ and is very easy to understand via a property called "strong approximation".

## 1.8 The Neighbor Method

We now describe the method of neighboring lattices, which is useful for enumerating all classes in a given (spinor) genus of quadratic forms. This can also be thought of as an analogous to the Hecke operators in the theory of modular forms.

**Definiton 1.8.1.** *Given two integer-valued quadratic lattice $L, L' \subset (V, Q)$ and some $m \in \mathbb{N}$, we say $L$ and $L'$ are $p$-**neighbors** for the prime number $p$ if $[L : L \cap L'] = [L' : L \cap L'] = m$ and $B(L, L') \notin \mathbb{Z}$.*

**Neighbor construction:** We can give an explicit construction of all $p$-neighbors in terms of vectors $\vec{v} \in L$. Given $L$, the sublattice $L \cap L'$ can be obtained by choosing some non-zero primitive vector $\vec{w} \in L$ and replacing $\vec{w}$ by $p\vec{w}$. Then the superlattice $L'$ of $L \cap L'$ is obtained by adjoining the vector $\frac{1}{p}\vec{v}$ for some vector $\vec{v} \in L \cap L'$. For this process to preserve the values of $B$, we must take $\vec{v}, \vec{w}$ so that $B(\vec{v}, \vec{w})$ is a unit.

The explicit construction of the neighboring lattices is given by starting with a fixed $\vec{v} \in L/pL$. We then look at the sublattice $L_v^0$ of $L$ defined by

$$L_{\vec{v}}^0 := \{\vec{x} \in L \mid B(\vec{v}, \vec{x}) \in p\mathbb{Z}\}$$

This is only a proper sublattice of $L$ when $\vec{v} \notin pL^{\#}$, because $\vec{v} \in pL^{\#} \iff B(\vec{v}, L) \subseteq p\mathbb{Z}$. To see $[L : L_{\vec{v}}^0] = p$ notice that over $\mathbb{Z}_p$ we can find a basis of $L$ whose first vector is $\vec{v}$, and whose second basis vector is $\vec{w}$ s.t. ... **FILL THIS IN!** (See [42, pp32–34] or [32, Defn 2.1, p739] for more details).

Therefore, the $p$-neighbors of $L$ can be described explicitly in the form

$$L' = L'_{\vec{v}} := L_{\vec{v}}^0 + \frac{1}{p}\mathbb{Z}\vec{v}$$

for certain vectors $\vec{v} \in L/pL$.

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An important fact about $p$-neighbors $L'$ of $L$ is that they are all in the same genus $\mathrm{Gen}(L)$. It is interesting to ask how many classes in the genus of $L$ can be created by taking repeated $p$-neighbors starting from $L$.

**Theorem 1.8.2.** *If $L$ is a $p$-neighbor of $L$ then $L' \in \mathrm{Gen}(L)$. If $p \nmid 2\det(L)$ and $n \geq 3$, then any $L'' \in \mathrm{Spn}(L)$ can be attained by taking repeated $p$-neighbors starting from $L$.*

*Proof.* This definition of $p$-neighbor and local equivalence of $p$-neighbors is proved in [42]. The spanning of the spinor genus is proved in [2]. $\square$

In fact, one can show that the transformation from $L$ to $L'$ can be attained with an element of spinor norm $p(\mathbb{Q}_p^\times)^2$. This is only well-defined up to an element of the spinor norms of the local isometries of $L$. Therefore we have that the spinor genus of the $p$-neighbors of $L$ lie in the spinor genus of $p \cdot \mathrm{Snorm}(L) \subseteq \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$, which agrees with the spinor genus of $L$ iff $p \in \mathrm{Snorm}(L)$. However, one can traverse the $p$-neighbor graph to enumerate the classes in $\mathrm{Spn}(L)$, and then choose another prime $q$ which allows one to move to another spinor genus and then again take $p$-neighbors. Repeating this process for different primes $q$, we can reach all spinor genera, and so we can enumerate all classes in $\mathrm{Gen}(L)$.

There is also a nice characterization of the $p$-neighbors on $L$ in terms of the non-singular points of the associated hypersurface $Q(\vec{x}) = 0$ over $\mathbb{F}_p$.

**Theorem 1.8.3.** *The $p$-neighbors of $L$ are in bijective correspondence with the non-singular points of $Q(\vec{x}) = 0$ in $\mathbb{P}^{n-1}(\mathbb{F}_p)$.*

*Proof.* See [42, Thrm 3.5, p34] or [32, Prop 2.2, p739]. $\square$

The $p$-neighbors can be organized into a weighted **$p$-neighbor graph** whose vertices are the classes in $\mathrm{Gen}(L)$, where two vertices are connected by an edge iff they are $p$-neighbors, and where the multiplicity of each edge is the number of distinct $p$-neighboring lattices of $L$ which are equivalent $\sim_{\mathbb{Z}}$ to $L'$. From the above theorem, we see that the $p$-neighbor graph is regular and that if $p \nmid 2\det(L)$ then the it is $p^{n-2}$-regular (i.e. every class has exactly $p^{n-2}$ neighbors for the prime $p$).

One can also use the above construction to define the $p$-neighbor operators $N_p$ on the free $\mathbb{Z}$-module $\mathcal{G} := \mathcal{G}(\mathbb{Z}) := \mathbb{Z}[\mathrm{Gen}(L)]$ generated by the classes $[L] := \mathrm{Cls}(L)$ in the genus of $L$. We define $N_p([L]) := \sum_{p\text{-neighbors } L' \text{ of } L} [L']$, and define its action on $\mathcal{G}$ by extending this linearly. One can also define a non-degenerate inner product $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ on $\mathcal{G}$ by

$$\langle [L], [L'] \rangle_{\mathcal{G}} := \begin{cases} |\{\alpha \in \mathrm{Aut}^+(L) \mid \mathrm{Snorm}(\alpha) \in (\mathbb{Q}_p^\times)^2\}| & \text{if } L \sim_{\mathbb{Z}} L', \\ 0 & \text{otherwise.} \end{cases}$$

Then we can check

**Theorem 1.8.4.** *The neighbor operators $T_p$ and $T_q$ commute for all primes $p, q \in \mathbb{N}$, and the $N_p$ are self-adjoint for $\langle \cdot, \cdot \rangle_{\mathcal{G}}$.*

*Proof.* See [42, Thrm 3.8, p39]. $\square$

This self-adjointness property means that one can find elements of $\mathcal{G}(\mathbb{C})$ which are simultaneous eigenfunctions of all $p$-neighbor operators $N_p$. It also means that they generate a semi-simple algebra of operators, and the associated neighbor module $\mathcal{G}$ is determined up to isomorphism by the traces of all $N_p$.

**Neighbor References:** Scharlau and Hemkemeier paper [32], Schulze-Pillot's paper for $n = 3$ and 4 [34], Tornaria's Thesis [42]. The neighbor method for Hermitian Forms is described in [33]. Other possible references are Eichler's classic book [10] (in German) and Gerstein's new book [15].

# Chapter 2

# Theta functions

## 2.1 Definitions and convergence

We now suppose that $Q$ is a positive definite quadratic form over $\mathbb{Z}$, in which case we know that the representation numbers $r_Q(m)$ are all finite, and zero if $m < 0$. This also tells us that $|\mathrm{Aut}(Q)| < \infty$ (by considering the action on the set of vectors with $Q(\vec{x}) \leq m$ once $m$ is large enough for this set to generate the lattice $\mathbb{Z}^n$). In this setting it makes sense to define the **theta function of** $Q$ as the Fourier series generating function for the representation numbers $r_Q(m)$ given by

$$\Theta_Q(z) := \sum_{m=0}^{\infty} r_Q(m) e^{2\pi i m z}.$$

Our main goal in this chapter is to understand the symmetries of this generating function very well, and to use them to obtain information about the original quadratic form $Q$.

To make $\Theta_Q(z)$ more than just a formal object, we should try to establish a some convergence properties so it can be regarded as an honest function. For this to converge (absolutely) we need the exponentials in the sum to be decaying, which happens if $z \in \mathbb{C}$ and $\mathrm{Im}(z) > 0$. For convenience, we denote by $\mathcal{H}$ the complex upper half-plane

$$\mathcal{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}.$$

The following theorem shows that any Fourier series with moderately (polynomially) growing coefficients will converge in $\mathcal{H}$.

**Theorem 2.1.1** (Convergence of Fourier series)**.** *The Fourier series*

$$f(z) := \sum_{m=0}^{\infty} a(m) e^{2\pi i m z}.$$

*converges absolutely and uniformly on compact subsets of* $\mathcal{H}$ *to a holomorphic function* $f : \mathcal{H} \to C$ *if all of its coefficients* $a(m) \in \mathbb{C}$ *satisfy* $|a(m)| \leq Cm^r$ *for some constant* $C > 0$.

*Proof.* Use Miyake, Koblitz or Diamond-Shurmann books for a proof reference... [][???] ☐

Because the number of lattice points in a smooth bounded region $\mathcal{R} \subset \mathbb{R}^n$ is approximately $\mathrm{Vol}(\mathcal{R})$, we see that $\sum_{i=0}^{M} r_Q(m) < CM^n$ for some constant $C$. This tells us that $r_Q(m) < C'M^{n-1}$ for some $C'$ and so the theorem above shows that the theta function converges to a holomorphic function when $z \in \mathbb{C}$ and $\mathrm{Im}(z) > 0$.

## 2.2 Symmetries of the theta function

While it is not obvious at first glance, $\Theta_Q(z)$ has a surprisingly large number of symmetries. From its definition as a Fourier series, it is clearly invariant under the transformation $z \mapsto z + 1$, but this is not particularly special since this holds for any Fourier series. However there is an additional symmetry provided to us by Fourier analysis because we can also see the theta function as a sum of a quadratic exponential function over a lattice $\mathbb{Z}^n$.

$$\Theta_Q(z) = \sum_{m=0}^{\infty} r_Q(m) e^{2\pi i m z} = \sum_{\vec{x} \in \mathbb{Z}^n} e^{2\pi i Q(\vec{x}) z}.$$

This additional lattice symmetry is realized through the Poisson summation theorem:

**Theorem 2.2.1** (Poisson Summation)**.** *Suppose that* $f(\vec{x})$ *is a function on* $\mathbb{R}^n$ *which decays faster than any polynomial as* $|\vec{x}| \to \infty$ *(i.e.* $\lim |x|^r f(\vec{x}) \to 0$ *as* $|\vec{x}| \to \infty$*). Then the equality*

$$\sum_{\vec{x} \in \mathbb{Z}^n} f(\vec{x}) = \sum_{\vec{x} \in \mathbb{Z}^n} \hat{f}(\vec{x})$$

*holds and the sums on both sides are absolutely convergent, where*

$$\hat{f}(\vec{x}) := \int_{y \in \mathbb{R}^n} f(\vec{y}) e^{-2\pi i \vec{x} \cdot \vec{y}} \, dy$$

*is the Fourier transform of* $f(\vec{x})$.

*Proof.* See [27, pp249-250] for a proof of this. ☐

The important point here is that the Gaussian function $f(x) = e^{-\pi \alpha x^2}$ transforms into a multiple of itself under the Fourier transform (which follows from checking $e^{-\pi x^2}$ is self-dual), so in the complex the $y$-part of each term will look like a decaying Gaussian (while the $x$-part will just oscillate), and so Poisson summation allows us to transform each term into itself after a little rescaling. This allows us to establish the following additional symmetry for the theta function under certain transformations of the form $z \mapsto \frac{-1}{Nz}$ for some $N \in \mathbb{N}$, giving:

**Theorem 2.2.2** (Symmetries of $\Theta_{x^2}(z)$)**.** *For all* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(4)$*, we have that*

$$\Theta_{x^2}\left(\frac{az+b}{cz+d}\right) = \left(\frac{-1}{d}\right)\sqrt{cz+d}\;\theta_{x^2}(z)$$

*where* $\frac{-\pi}{2} < \arg(\sqrt{z}) \leq \frac{\pi}{2}$ *and* $\left(\frac{-1}{d}\right) := (-1)^{\frac{d-1}{2}}$.

*Proof.* This is stated on [37, p440]. See also Knopp [22] and Andrianov-Zhuralev [1] for proofs. ☐

**Theorem 2.2.3** (Symmetries of $\Theta_Q(z)$ for general $Q$)**.** *Insert symmetries here! =)*

A nice treatment of transformation formulas for theta series in an even number of variables along these lines is given in the appendix to Chapter 1 of Eichler's book [9, pp44-52]. See also Knopp's book [22, Ch3, §4, pp45–48] for transformation laws for $n = 1$. The general computation can be found (in much greater generality) in [1, ???]

## 2.3 Computation of Gauss sums for establishing symmetries

Here we give an exact formula for the theta series $\Theta_{x^2}(z)$ as it approaches a rational point from $\mathcal{H}$ in terms of Gauss sums. This is done explicitly in [9, pp46-48]. This computation also explains why the singular series used in the "circle method" involves Gauss sums, and also illustrates the local and multiplicative nature of the boundary values.

## 2.4 Modular Forms

It is useful to try to understand these symmetries in terms of certain standard kinds of groups acting by linear fractional transformations on $\mathcal{H}$. One way to do this is to compare them to the **congruence subgroups** $\Gamma_0(N)$, which are defined as the elements of $\mathrm{SL}_2(\mathbb{Z})$ which look upper triangular mod $N$. In this way we obtain a more standard presentation of the symmetries of $\Theta_Q(z)$ in terms of simpler groups which are easier to understand, but at the expense of a few symmetries (which distinguish the theta group from $\Gamma_0(N)$).

**Theorem 2.4.1.** *... describe the theta group and its symmetries for* $\Theta_Q(z)$*...*

We define a **modular form** of weight $k$, level $N$, and (mod $N$ Dirichlet) character $\chi$ to be a holomorphic function on the complex upper half plane $\mathcal{H}$ which transforms with respect to the group

$$\Gamma_0(N) := \left\{\gamma := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N}\right\}$$

under the rule

$$f\left(\frac{az+b}{cz+d}\right) = \varepsilon(\gamma, k)\chi(d)(cz+d)^k f(z)$$

for all $\gamma := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$, and satisfies the additional technical condition that $f(z)$ is also "holomorphic" at the boundary values $\mathbb{P}^1(\mathbb{Q}) := \mathbb{Q} \cup \{\infty\}$ of the quotient $\Gamma_0(N)\backslash\mathcal{H}$. When $k \in \mathbb{Z}$ we take the multiplier system $\varepsilon(\gamma, k) \cong 1$, and otherwise when $k \in \frac{1}{2}\mathbb{Z} - \mathbb{Z}$ we take $\varepsilon(\gamma, k)$ to be the theta multiplier system (described in detail in [22, Ch 4] or [?]). Good references for the theory of general theory of modular forms are [20, 4, 29, 23, 38]. In this language, the above theorem can be restated as saying that

**Corollary 2.4.2.** $\Theta_Q(z)$ *is a modular form of weight* $\frac{n}{2}$ *and level* $N$ *and character* $\chi(\cdot) = \left(\frac{(-1)^{\lfloor \frac{n}{2} \rfloor}\det(Q)}{\cdot}\right)$ *(taken with respect to the theta multiplier system when* $n$ *is odd).*

Note that here the term "level" refers both the level of the quadratic form as well as the level of the modular form (i.e. we use symmetries from $\Gamma_0(N)$).

**Facts about Modular forms:** There are several main structural results in the theory of modular forms that are useful for understanding theta series, which we mention now:

1. The space of modular forms $M_k(N, \chi)$ with fixed invariants $(k, N, \chi)$ is a finite dimensional vector space over $\mathbb{C}$.

2. The space $M_k(N, \chi)$ can be decomposed uniquely as a direct sum of cusp forms (of functions vanishing at all cusps) and Eisenstein series (spanned by the Eisenstein series associated to the cusps of $\Gamma_0(N)$).

3. Any function in the Eisenstein space has Fourier coefficients $a_E(m)$ which are about as large as $m^{k-1}$ when they are not zero.

4. Any cusp form has Fourier coefficients $a_f(m)$ which are (trivially) no larger than $Cm^{\frac{k}{2}}$.

**(Warning: We have omitted all factors involving** $\log(m)$ **for simplicity above.)**

These facts have the consequence that theta series Fourier coefficients $r_Q(m)$ (which are always $\geq 0$) are asymptotically controlled by the Eisenstein series coefficients, and so are about as large as $m^{k-1}$ as $m \to \infty$ when $n \geq 5$. For $n \leq 4$, the above results are not enough to show that the Eisenstein coefficients are asymptotically larger than the cusp form coefficients, so more care is needed. The case $n = 4$ was first handled by Kloosterman by a clever refinement of the Circle Method (described briefly below), and has since been absorbed into the theory of modular forms as a consequence of the Ramanujan bound $|a_f(p)| \leq 2\sqrt{p}$ for prime coefficients of weight 2 cusp forms. This case also involves additional local considerations at finitely many primes $p$ where $Q$ is anisotropic over $\mathbb{Q}_p$.

The case $n = 3$ is even more delicate, and involves additional arithmetic and analytic tools to understand (e.g. spinor genera, the Shimura lifting of half-integral weight forms, analytic bounds on square-free coefficients of half-integral weight forms). In particular it was handled by Duke and Schulze-Pillot, and then by Schulze-Pillot in the papers [6, 35] For more details on asymptotic results, please see the survey papers [18, 5, 19, 36]. The case of binary forms is a genuinely arithmetic problem (since for weight $k = 1$ both cusp forms and Eisenstein series coefficients do not grow differently as $m \to \infty$), and it exhibits a much closer connection to explicit class field theory for quadratic extensions than the asymptotic results described here.

## 2.5 The circle method and Siegel's Formula

The origins of the many of the modern analytic techniques in the theory of quadratic forms have their origins in the famous "circle method" of Hardy, Littlewood and Ramanujan. The idea of this method is that one can express the number of representations $r_Q(m)$ for $Q = a_1x_1^2 + \cdots + a_nx_n^2$ in as an integral over the unit circle which can be well-approximated by taking small intervals about angles which are rational multiples of $2\pi$ (where small here means small relative to the overall denominator of the fractional multiples one considers). These rational angle contributions can be thought of locally, and so we learn that local considerations give a good approximation of the number of representations $r_Q(m)$ for $Q = a_1x_1^2 + \cdots + a_nx_n^2$ when $n$ is large enough. In the language of modular forms this method produces an Eisenstein series $E_Q(z)$ (called a "singular series") with multiplicative Fourier coefficients that agrees with the theta series $\theta_Q(z)$ at all rational points (and at $\infty$) so the difference $\theta_Q(z) - E_Q(z)$ is a cusp form and so must have asymptotically smaller Fourier coefficients than $E(z)$. This cusp form can be analyzed to various degrees, but the most naive bound for its Fourier coefficients gives non-trivial asymptotic information for the asymptotic behavior of $r_Q(m)$ for $m \geq 5$. (See [30, Ch 6, pp ???] and [22, Ch 5, pp63–87] for details.) The case $n = 4$ can also be handled, but requires an essential refinement of Kloosterman to obtain additional cancellation. (See [20, §11.4–5, pp190–199] and [21, ???].) for more details.)

Siegel used these ideas to give quantitative meaning to the Fourier coefficients in the singular series both in terms of the underlying space of modular forms (as an Eisenstein series), but also in terms of the "local densities" associated to the quadratic form $Q$. In particular he proved the following theorem:

**Theorem 2.5.1** (Siegel). *Suppose $Q(\vec{x})$ is a positive definite integer-valued quadratic form in $n \geq 5$ variables, whose theta function $\theta_Q(z)$ is written (uniquely) as a sum of an Eisenstien series $E(z)$ and a cusp form $f(z)$. Then $E(z) = \sum_{m \geq 0} a_E(m) e^{2\pi i m z}$ can be expressed in two different ways:*

*Firstly, $E(z)$ a weighted sum over the genus of $Q$:*

$$1) \qquad E(z) = \frac{\sum_{Q' \in \mathrm{Gen}(Q)} \frac{\theta_{Q'}(z)}{|\mathrm{Aut}(Q')|}}{\sum_{Q' \in \mathrm{Gen}(Q)} \frac{1}{|\mathrm{Aut}(Q')|}}$$

*showing that $E(z)$ is a genus invariant.*

*Secondly, the Fourier coefficients $a_E(m)$ can be expressed as an infinite local product*

$$2) \qquad a_E(m) = \prod_{places\, v} \beta_{Q,v}(m)$$

*where the numbers $\beta_{Q,v}(m)$ are the* **local representation densities of $Q$ at** *$m$, defined by the limit*

$$\beta_{Q,v}(m) := \lim_{U \supset_{open} \{m\}, U \to \{m\}} \frac{\mathrm{Vol}(Q^{-1}(U))}{\mathrm{Vol}(U)}$$

*with respect to the natural volumes on $n$-dimensional and 1-dimensional affine space.*

*Proof.* See Siegel's Lecture notes [40] or his orginal series of papers []. □

These formulas are extremely important for the analytical theory of quadratic forms, and can be used to provide precise asymptotics for $r_Q(m)$ as $m \to \infty$. Extensions of this technique led Siegel to prove analogous results for more general kinds of theta functions which count representations of a quadratic form by another quadratic form. These are examples of "Siegel modular forms" which have analogous symmetries for the symplectic group $\mathrm{Sp}_{2r}$. (See [1] for more details.)

The formulas of Siegel were later generalized by Weil to a more representation-theoretic context by means of a certain very simple representation of a sympelctic group called the "Weil representation" that we will meet later. This representation can be used to give a proof of Siegel's formulas in the case where $Q$ is a positive definite quadratic form in $n \geq 5$ variables. It is interesting to see the progression of ideas from the circle method to modular forms to the Weil representation, and to notice that while the language used to obtain these results changes to suit our deepening perspective and context, that the essential features (and technical difficulties) of the result remain very much the same.

## 2.6 Mass Formulas

Describe the mass formula, and its connection with Siegel's formula, Tamagawa numbers, and class numbers.

## 2.7 Some results using modularity

1. Jacobi's theorem for $x^2 + y^2 + z^2 + w^2$
2. Sums of squares and exact formulas
3. Ramanujan's ternary form (Sound/Ono)
4. Duke/SP and local global results for ternary forms and equidistribution
5. Computational bounds and 290-Theorem
6. The Basis Problem for $\Gamma_0(N)$

# Chapter 3

# Quaternion Algebras

Quadratic forms are closely connected with quadratic extensions, both those which are commutative (quadratic fields and their rings of integers) and non-commutative (quaternion algebras and their maximal orders). In this chapter we explore some connections with non-commutative algebras of a particularly nice kind (known as "central simple algebras"), and describe their basic structure. Good references for central simple algebras are the books [31, 16].

## 3.1 Definitions

We begin by defining a **central simple algebra** $A$ as a finite-dimensional (possibly non-commutative) algebra over a field $k$ whose center is $k$ and which contains no proper non-zero two-sided ideals. To make the dependence on $k$ explicit, we sometimes write $A$ as $A/k$. We say that the **dimension** of $A/k$ is the dimension of $A$ as a vector space over $k$.

**Theorem 3.1.1.** *Suppose that $A_1$ and $A_2$ are central simple algebras over $k$. Then the tensor product $A_1 \otimes_k A_2$ is also a central simple algebra over $k$.*

Another nice property of central simple algebras is that we can freely extend the base field $k$ and preserve the property of being central simple (though now with a larger center!):

**Theorem 3.1.2.** *Suppose that $A$ is a central simple algebras over $k$, and $K$ is a field containing $k$. Then $A \otimes_k K$ is a central simple algebra over $K$.*

The simplest examples of central simple algebras are the matrix algebras $M_n(k)$ (which have dimension $n^2$). Notice that any central simple algebra over $k$ which is commutative must be just $k$ itself, so in general central simple algebras are non-commutative. The next simplest example of a central simple algebra which is not a field (i.e. non-commutative) is called a **quaternion algebra**, and can be defined in terms of a basis $\mathcal{B} = \{1, i, j, k\}$ satisfying the relations $i^2 = a$, $j^2 = b$, $k := ij = -ji$ for some fixed $a, b \in k^\times$ (where we always assume that $\mathrm{char}(k) \neq 2$). This quaternion algebra is often referred to by the symbol $\left(\frac{a,b}{k}\right)$, though various different choices of $a$ and $b$ may give rise to isomorphic quaternion algebras (e.g. $\left(\frac{1,-1}{k}\right) \cong \left(\frac{4,-1}{k}\right)$).

If $A \cong M_n(k)$ for some $n$, we say that $A$ is **split**. If it happens that $A_K := A \otimes_k K$ is split for some extension $K$ of $k$, we say that $A_k$ is **split by** $K$, or that $K$ is a **splitting field** for $A/k$. The following theorem allows us to use this idea to reduce to analyzing the split case:

**Theorem 3.1.3.** *If $A_k$ is a central simple algebra over $k$, then $A_k$ is split by some finite extension $K/k$.*

Since base change doesn't change the degree of a central simple algebra, and we can always change base so that it is a matrix algebra, we have the following useful corollary:

**Corollary 3.1.4.** *The degree of a central simple algebra is always a square.*

We can use this idea to define a norm map from $A$ to $k$ by first extending scalars to a splitting field $\bar{k}$, giving an isomorphism $A \cong M_n(D)$ defined over $\bar{k}$ by Wedderburn's theorem. However there are no non-trivial division algebras over $\bar{k}$, so $A \cong M_n(\bar{k})$. We then define the **norm** $N_{A/k}(x)$ as the determinant of $x$ under this isomorphism. Since $\det(x)$ is constant on conjugacy classes, the norm is independent of the choice of isomorphism, and is invariant under the Galois action as well, hence in $k$. Since the determinant is multiplicative, we see that

$$N_{A/k}(ab) = N_{A/k}(a)N_{A/k}(b) \qquad \text{for all } a, b \in A.$$

If it happens that every non-zero element of $A$ is invertible (i.e. $a \in A - \{0\} \implies \exists a' \in A$ so that $aa' = 1$ and $a'a = 1$) we say that $A$ is a **division algebra**. One can think of division algebras as natural non-commutative generalizations of a (finite degree) field extensions $K/k$. In fact any non-zero element $a$ of a central simple algebra $A$ of degree $n^2$ generates a commutative subalgebra $k(a) \subset A$ of dimension $\leq n$. In the case of a quaternion algebra one can realize the norm map in terms of a conjugation operation explicitly on the basis (by taking $\alpha = a + bi + cj + dk \mapsto \bar{\alpha} := a - bi - cj - dk$), giving the norm as $N_{A/k}(\alpha) = \alpha\bar{\alpha}$. The property of being a division algebra can be easily characterized in terms of the norm map.

**Theorem 3.1.5.** *A central simple algebra $A$ is a division algebra iff the condition $N_{A/k}(a) = 0 \iff a = 0$ holds.*

**Corollary 3.1.6.** *A quaternion algebra $A$ is a division algebra iff the quadratic norm form $N_{A/k}(\vec{x})$ is anisotropic.*

The fundamental structural result in the theory of central simple algebras is the following theorem of Wedderburn:

**Theorem 3.1.7** (Wedderburn)**.** *Every central simple algebra $A_k$ over $k$ is isomorphic to a tensor product of matrix rings over division algebras, i.e.*

$$A_k \cong \otimes_{i=1}^{r} M_{n_i}(D_i)$$

*where $D_i$ are division algebras over $k$, and $n_i \in \mathbb{N}$.*

Specializing to quaternion algebras, we have the following useful local classification result:

**Theorem 3.1.8.** *There are exactly two quaternion algebras (up to isomorphism) over the local fields $\mathbb{Q}_p$ and $\mathbb{R}$.*

Over $k = \mathbb{Q}_p$ or $\mathbb{R}$, one of these quaternion algebras is the split $M_2(k)$. Accordingly, we say that $A$ is either **split** (if $A \cong M_2(k)$) or **ramified** (otherwise). Since any isotropic vector of the norm form $N_{A/k}(\cdot)$ ensures that $A$ is split, we see that the other quaternion algebra must have anisotropic norm form, which gives a construction of the unique 4-dimensional anisotropic form (up to equivalence) over $\mathbb{Q}_p$. If the quaternion algebra $A$ is given by the relations $(a, b)$ for $a, b \in \mathbb{Q}_v$, then the Hilbert symbol $(a, b)_v$ describes whether $A$ is split or ramified. More precisely,

**Theorem 3.1.9.** *A local quaternion algebra $A/\mathbb{Q}_v$ defined by the relations $(a, b)$ for $a, b \in \mathbb{Q}_v$ is split $\iff$ the Hilbert symbol $(a, b)_v = 1$.*

Given a (global) quaternion algebra $A/\mathbb{Q}$, we can consider the associated (local) quaternion algebras $A_{\mathbb{Q}_v}$, each of which is either split or ramified. From the definition of Hilbert symbols, we see that $A_v$ is split at all but finitely many places $v$, and the product formula for Hilbert symbols

$$a, b \in \mathbb{Q}^\times \implies \prod_v (a, b)_v = 1$$

gives that $A$ is ramified at an even number of places $v$.

## 3.2 The Clifford Algebra

This is described in the books of Cassels [3, pp169–175], O'Meara, Knus, Lam, ...

Give basic definition over fields and rings which are PID's.

The classification of Clifford algebras of small dimensional quadratic forms is done in [].

Examples for forms of degree 1, 2, 3, 4:

1. Degree 1 => Quadratic Field
2. Degree 2 => Composition law from the even part
3. Degree 3 => Quaternion algebra from the even part
4. Degree 4 => $SO(4) = H \times H$ for some quaternion algebra $H$ (which is split?)

## 3.3 The Spin Group

See [3, pp175–182] for details.

1. Constructing the Spin group
2. The Spinor Norm map
3. Strong Approximation
4. Consequences for class numbers

## 3.4 Relation to Quadratic Forms locally

Description of the relation between quaternion algebras and the maximal anisotropic subspace of a quadratic form.

1. Quaternion algebra -> QF via norm
2. QF -> Quaternion algebra (via anisotropic subspace)

## 3.5 Orders and Ideals

See the thesis of Voight [43], Kohel's paper [25] and Eichler's article [7, 8] for details.

1. Orders and Maximal orders
2. Norm forms
3. Two-sided and Left ideals
4. Class Number and Type Numbers
5. Define Typ($\mathcal{O}$) the classes of lattices which are locally

## 3.6 Ternary forms and Quaternion orders

This is described from the perspective of supersingular elliptic curves and Mestré's method of graphs in [24].

In this section we describe a bijective correspondence between classes of ternary quadratic forms over $\mathbb{Z}$ and types of orders in an associated quaternion algebra $B$. Explicitly, given a ternary quadratic form $Q(\vec{x})$ over $\mathbb{Z}$ we know that its rational even Clifford algebra $\mathcal{C}_{\mathbb{Q}}^{+}(Q)$ is 4-dimensional with center $\mathbb{Q}$, hence $\mathcal{C}_{\mathbb{Q}}^{+}(Q)$ is a quaternion algebra. If we let $B := \mathcal{C}_{\mathbb{Q}}^{+}(Q)$, then $Q$ gives rise to the quaternion order $\mathcal{O}_Q := \mathcal{C}_{\mathbb{Z}}^{+}(Q) \subset B$.

It is more convenient to talk about lattices here than quadratic forms, so we associate $Q$ with the lattice $L$, and choose lattices $L_i \subset (V, Q)$ representing the forms $Q_i \in \mathrm{Gen}(Q) = \mathrm{Gen}(L)$. For any lattice $L' \subset (V, Q)$ we can consider its associated quaternion order $\mathcal{O}_{L'} := \mathcal{C}^{+}_{\mathbb{Z}}(L') \subset B$.

If $L'$ is locally equivalent to $L''$, then the universal mapping property of clifford algebras (**Cross-reference this!**) gives an induced ring isomorphism on the associated orders, and by the Skolem-Noether theorem we know this arises as an inner automorphism by conjugation from $B$, so we have a well-defined map from the classes in $\mathrm{Gen}(L)$ to the classes of orders in $\mathrm{Typ}(\mathcal{O}_L)$.

**Some interesting points:**

1. Note: This also gives the correspondence between unary forms and quadratic orders.
2. Question: What can be said about the connection between octonian orders and 5-dim'l quadratic forms?

Conversely we can recover a ternary quadratic form from quaternion order by looking at the negative discrimant form $-\Delta(\vec{x}) := 4\mathrm{N}_{B/\mathbb{Q}}(\vec{x}) - (\mathrm{Tr}_{B/\mathbb{Q}}(\vec{x}))^2$ (given as the negative of the discriminant of the quadratic characteristic polynomial $P(\vec{x})$ of $\vec{x}$ in $B$). This is a singular form because the polynomial associated to the scalars $\mathbb{Q}$ have a double root. The restriction of $-\Delta(\vec{x})$ to $\mathcal{O}$ visibly splits as a sum of the scalar space and some other lattice and a short computation checks that with respect to some choice of basis, the $3 \times 3$ submatrix of the restricted form is the adjoint of the original quadratic form, and we have a natural duality between the even and odd graded parts in the Clifford algebra. To see this we need to check on a basis that the quadratic form on the odd part is the adjoint of the form on the even part. This computation also shows that if we then take the scaled adjoint of the dual of this quaternion order, we recover the original quadratic form. Thus we have a bijection between lattices and orders, which then descends to an equivalence between classes of quadratic forms and types of quaternion orders, giving

**Theorem 3.6.1.** *There is a bijection between the classes of (non-degenerate) ternary quadratic forms in a genus* $\mathrm{Gen}(T)$ *and the classes of types* $\mathrm{Typ}(\mathcal{O}_T)$ *of the associated quaternion order* $\mathcal{O}_T$*.*

## 3.7 Brandt and Neighbor Matrices

See [25] and [7, 8] for details.

State definitions of Brandt matrices (in terms of left ideal classes) and neighbor (Anzahl) matrices.

**Theorem 3.7.1** (Eichler Commutation relations)**.**

This shows that the natural map associating ideals to their right orders gives a ring isomorphism from the Brandt matrices to the Anzahl matrices. The kernel of this map can be described explicitly in terms of the sign of the functional equation on the level modular forms on $\mathrm{SL}_2$.

The Brandt matrices for $B_{p,\infty}$ acting on ideal classes can be shown to give an isomorphism with the Hecke operators acting on weight 2 modular forms on $\Gamma_0(p)$. This is proved by Eichler [7] by showing that the Brandt matrices and Hecke operators act with the same traces, and since they are both semisimple modules, they must be isomorphic. This is also proved by Gross in [17] by more geometric methods.

**Theorem 3.7.2** (Eichler). *The Hecke module $M_2(p)$ is isomorphic to the Brandt module of linear combinations of ideal classes of $B_{p,\infty}$.*

*Sketch of Proof.* The map is given explicitly by associating to any pair of ideal classes $(I_i, I_j)$ the normalized theta series $\theta_{i,j}$ of the lattice $I := I_j^{-1} I_i$ given as the theta series of the primitive norm form $\frac{1}{\mathrm{N}_{B/\mathbb{Q}}(I)} \mathrm{N}_{B/\mathbb{Q}}(\vec{x})|_I$. □

## 3.8 Supersingular elliptic curves

See Silverman's book [41, Ch V, §3–4] for basic facts about supersingular elliptic curves and [24] and [7, 8] for their connection with quaternion orders.

## 3.9 Other results

1. Action of the class group of $\mathbb{Q}(\sqrt{-D})$ on vectors of length $D$.
2. Two composition laws papers ($n = 2, 4$)
3. Class number of Eichler order = dimension of weight 2 modular forms

# Chapter 4

# Liftings

## 4.1 Liftings of modular forms

The following liftings can be described explicitly in terms of the Weil representation

1. The Shimura lift
2. The Doi Naganuma lift
3. The Jacquet-Langlands lift

though we will only describe the Shimura lift in these notes. See the small book of Friedberg [11] for more details.

## 4.2 Classical to Adelic modular forms for $\mathrm{GL}_2$

It is possible to interpret the transformation property of modular forms in a simpler way by reinterpreting them as functions on the group $\mathrm{GL}_2$. We do this in two steps, first by lifting $f(z)$ to a function on $\mathrm{GL}_2(\mathbb{R})$, and then by lifting to a function on the adelic group $\mathrm{GL}_2(\mathbb{A})$ where the transformation properties can be seen most simply. This adelic perspective will also give us a simple language to use to describe the lifting of modular forms via the Weil representation. This passage from classical to adelic modular forms is described in [13, §3] and [39, §10].

Given a modular form $f(z) : \mathcal{H} \to \mathbb{C}$ as in (**??**), we can associate a function $\tilde{f}(g) : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C}$ by lifting the action of $\mathrm{GL}_2(\mathbb{R})$ on the point $i \in \mathcal{H}$. By using the transitive action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathcal{H}$ determined by its action on any given point in $\mathcal{H}$ (say $i \in \mathcal{H}$), we define

$$\tilde{f}(g) := f(g \cdot i)(cz + d)^{-k}\chi(d)^{-1} = f \mid_k (g)$$

We can then lift $\tilde{f}(g)$ to an adelic function $F : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ by viewing the "untwisting" of the transformation factors $(cz + d)^{-k}\chi(d)^{-1}$ locally, with the $(cz + d)^{-k}$ coming from the real place, and writing the Dirichlet character $\chi(d)^{-1}$ as a product of local characters $\chi(d)_p^{-1} : \mathbb{Z}_p \to \mathbb{C}$.

## 4.3 Adelic modular forms for general groups

**References for this section:** For adelizations of algebraic groups and modular forms for an adelic group see [39, §8, 10, 11], for definitions of adelic modular forms on $\mathrm{GL}_2$ [13, §1.3, pp40-53], for adelic modular forms on a quaternion algebra see Hida's book [].

In this section we give a general definition for adelic modular forms for a general algebraic group $G$. This agrees with the definitions above when $G = \mathrm{GL}_2$, and in future sections we will want to consider $G$ to be either a special orthogonal group $SO(Q)$ or the multiplicative group $B^\times$ of a quaternion algebra.

We first define the adelization of an affine/linear algebraic group $G$ over the ring of integers $\mathcal{O}$ of a number field $F$ (defined as the zero set of an ideal of relations in a polynomial ring $R[\vec{x}]$) to be the **restricted direct product** $\prod_v' G(F_v)$ of the local algebraic groups $G(F_v)$ over all places $v$ of $F$, which is the subset of the usual direct product $\prod_v G(F_v)$ satisfying the restriction that $g_\mathbb{A} = (g_v)_v$ is subject to the restriction that $g_v \in G(\mathcal{O}_v)$ for all but finitely many $v$. The restricted direct product has several advantages over the usual direct product – it is small enough to be locally compact because every element has all but finitely many components in the compact group $G(\mathcal{O}_v)$, and it is large enough to contain all rational points $G(F)$.

For the convenience of the reader, we will consistently use subscripts (e.g. $\mathbb{A}, v, \mathbf{a}, \mathbf{f}$) to denote the kind of element (resp. adelic, local, archimedean, non-archimendian) that the element $g_. \in G(F_.)$ represents. We also denote the center of $G$ by $Z$, to which the same conventions apply for $z \in \mathbb{Z}$. Elements without subscripts will represent rational elements (i.e. we take $g \in G(F)$). In most cases $Z(F) = F^\times$ and $Z_\mathbb{A} := Z(F_\mathbb{A})$ are the ideles of $F$. It is also common to describe the compact groups $G(\mathcal{O}_v)$ as $K_v$, with $K_v$ denoting a fixed choice of the maximal compact subgroup $G(K_v)$ in when $v$ is archimedean.

Now we say that a function $F : G_\mathbb{A} \to \mathbb{C}$ is an **adelic automorphic form** for $G$ if

1. $F$ is left invariant for the rational group: $F(g \cdot g_\mathbb{A}) = F(g_\mathbb{A})$ for all $g \in G(F)$ and for all $g_\mathbb{A} \in G_\mathbb{A}$.

2. $F$ has an adelic central character $\psi : Z_\mathbb{A} : Z(\mathbb{A}) \to \mathbb{C}$ so that $F(z_\mathbb{A} \cdot g_\mathbb{A}) = F(g_\mathbb{A})\psi(z_\mathbb{A})$ for all $z_\mathbb{A} \in Z_\mathbb{A}$ and for all $g_\mathbb{A} \in G_\mathbb{A}$.

3. $F$ is right "$K_\mathbb{A}$-finite", meaning that $F$ transforms by right multiplication the compact group $\prod_{v\in\mathbf{f}} G(\mathcal{O}_v)$ by some finite dimensional adelic representation $\lambda_\mathbb{A}$: $F(g_\mathbb{A} \cdot k_\mathbf{f}) = (\lambda(k_\mathbb{A})F)(g_\mathbb{A})$ for all $k_\mathbb{A} \in K_\mathbb{A}$ and for all $g_\mathbb{A} \in G_\mathbb{A}$.

4. $F_\mathbf{a}$ is smooth and "$\mathfrak{z}_\mathbf{a}$-finite", where $\mathfrak{z}_\mathbf{a}$ is the center of the universal enveloping algebra for $\mathbb{G}_\mathbf{a}$: meaning that the image of $F$ under $\mathfrak{z}$ spans a finite-dimensional vector space.

We note that $\mathfrak{z}$ can also be interpreted as the ring of bi-invariant differential operators on $\mathbb{G}_{\mathbf{a}}$, and in the case of $\mathrm{GL}_2(\mathbb{R})$, $\mathfrak{z}_{\mathbf{a}}$ is the ring $\mathbb{C}[\Delta]$ where $\Delta$ is the hyperbolic Laplacian operation $-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$.

5. $F$ has moderate growth: meaning that there are real constants $C$ and $M$ so that $|F(d_{\mathbb{A}}g_{\mathbb{A}})| \leq C|a|^M$ for all diagonal elements $d_{\mathbb{A}} \in ?????$ and all $g_{\mathbb{A}}$ in any fixed compact subset of $G_{\mathbb{A}}$.

The most important conditions for us will be conditions (1)-(3). Condition (4) is a generalization of the usual holomorphy condtition for classical modular forms (since any real analytic function is eigenfunction of $\Delta$ with eigenvalue zero), and condition (5) is a technical growth condition used to exclude poorly behaved functions like **give example!**. In the case where $G$ is an orthogonal group or a quaternion algebra conditions (4) and (5) can be safely omitted because the archimedean component is already compact or becomes so in the quotient without the addition of "cusps". **Interesting Examples to Discuss:** - on $\mathrm{GL}_2$ (coming from $\mathcal{H}$ and $SL_2(\mathbb{R})$)

- on quaternion algebras $H$ (also mention Brandt matrices)
- on $\mathrm{O}(Q)$ where $Q$ is definite and indefinite

## 4.4 Hecke Algebras for adelic groups

See Shimura's book [39, §11] and Gross's paper

## 4.5 The Weil representation

To see how theta functions arise in terms of representation theory, we now define the **Weil representation** whose symmetries will be closely related to the Fourier transform. We will not go through the explicit construction of the Weil representation, but instead content ourselves to list its defining properties below and go on to use the Weil representation to produce classical theta functions. A good explicit references for constructing the Weil representation are [28], or Gelbart's book [14]. Other places where the Weil representation are used in a similar way are [**?**, **?**], Kudla's lecture notes [26], and Gelbart's book, [12, §7A, pp134–150].

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These considerations give rise to the adelic Weil Representation of $\mathcal{W} : \mathrm{Sp}_2(F)\backslash\mathrm{Sp}_2(\mathbb{A}) \to \mathrm{GL}(S(V_{\mathbb{A}}))$ which is defined by the following properties:

1. $\left(\mathcal{W}\left(\begin{bmatrix} a & 0 \\ 0 & {}^t a^{-1}\end{bmatrix}\right)\Phi\right)(\vec{v}) = \chi_V(a)|\det(a)|_{\mathbb{A}}^{\frac{n}{2}}\Phi({}^t a\vec{v}) \quad \text{where } \chi_V(\cdot) = (\cdot, (-1)^{n/2}\det(Q))_{\mathbb{Q}}$

2. $\left(\mathcal{W}\left(\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}\right)\Phi\right)(\vec{v}) = e^{2\pi i x Q(\vec{v})}\Phi(v)$

3. $\left(\mathcal{W}\left(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right)\Phi\right)(\vec{v}) = \gamma\hat{\Phi}(v)$ where $\gamma$ is some eighth root of unity.

An explicit reference for these formulas are [14, Thrm 2.22, p37], and [26, I.1.6, p3] (though there is a minor typo writing $x$ for $a$ in the second formula there). These formulas uniquely define the Weil representation, since any element of $\mathrm{SL}_2$ can be expressed as a product of these elements.

## 4.6 Theta kernels and theta liftings

The Weil representation restricted to our pair $\mathrm{Sp}(W)\times O(V) =: G\times H$ incorporates both an invariance under the orthogonal group, and a Fourier transform from the Weyl element. To produce theta functions from this we will need to introduce the familiar classical features of a self-dual function on a lattice. In the adelic context, our lattice is provided by the rational points $V(F)$, and the adelic self-dual function is taken to be the familiar Gaussian exponential at $\infty$ and the characteristic function of some fixed lattice $L$ on $V$ elsewhere.

For convenience, we fix the adelic function $\phi_{\mathbb{A}}(\vec{v}_{\mathbb{A}}) = \prod_v \phi_v(\vec{v}_v)$ where $\phi_\infty(\vec{v}) = e^{-\pi Q(\vec{v})}$ and $\phi_v(\vec{v}) =$ characteristic function of $\mathbb{Z}_p^n$. Then we define the **theta kernel**

$$\theta_\phi(g,h) := \sum_{\vec{v}\in V(F)} (\mathcal{W}(g,h)\phi)(\vec{v}) = \sum_{\vec{v}\in V(F)} (\mathcal{W}(g)\phi)(h^{-1}\vec{v})$$

which averages the function $\phi(\vec{v})$ over the rational lattice $V(F)$. This lattice average resembles the usual theta function in its construction, though in a different context. In particular, here Poisson summation tells us that the theta kernel has the rational lattice symmetries of being left-invariant under $\mathrm{Sp}(W)$. It is also left-invariant under $O(V(F))$ because that action just permutes $V(F)$. This tells us that

**Lemma 4.6.1.** *The theta kernel* $\theta_\phi(g,h)$ *is a function on* $\mathrm{Sp}(W)\backslash\mathrm{Sp}(W_{\mathbb{A}})\times O(V)\backslash O(V_{\mathbb{A}})$.

This rational bi-invariance is exactly what allows us to use the theta kernel to move automorphic forms between the symplectic and orthogonal groups. Suppose we have an automorphic form $F(h_{\mathbb{A}})$ on the orthogonal group $O(V_{\mathbb{A}})$. Then we can define its **theta lift** by the averaging

$$\Theta(F)(g_{\mathbb{A}}) := \int_{h_{\mathbb{A}}\in O(V)\backslash O(V_{\mathbb{A}})} F(g_{\mathbb{A}})\,\theta_\phi(g_{\mathbb{A}},h_{\mathbb{A}})\,dh_{\mathbb{A}}$$

of $F$ against the theta kernel with respect to the choice of adelic Haar measure $dh_{\mathbb{A}}$ on $\mathcal{O}(V_{\mathbb{A}})$ giving $\mathrm{Stab}_{\mathbb{A}}(L)\subset O(V_{\mathbb{A}})$ volume 1. The theta lift $\Theta(F)(g_{\mathbb{A}})$ formally inherits the symplectic invariance of the theta kernel, and gives an automorphic form on $\mathrm{Sp}(W_{\mathbb{A}})$ when this integral converges.

We will not establish the invariance under an open compact subgroups $K_{\mathbb{A}}$, however this will follow from the classical theta function transformation properties for our example.

## 4.7 Theta liftings

The symmetries of the Weil representation can be used to construct an explicit lifting of automorphic forms between different groups. This lifting will be constructed using a high-brow version of the same familiar symmetries that were used to prove the transformation properties of the classical $\theta$-function in §2.

We can use the theta kernel to move automorphic forms between the symplectic and orthogonal groups.

## 4.8 Computing a simple lift: $\Theta_{ax^2}(z)$

We now compute the theta lifting of the characteristic function $\Phi_L$ of the coset of $O_{\mathbb{A}}$ corresponding to our chosen quadratic lattice, with respect to the fixed choice of function $\phi_{\mathbb{A}}(\vec{v}_{\mathbb{A}})$ described above.

This gives

$$\begin{aligned} \Theta(\Phi_L)(g_{\mathbb{A}}) &= \int_{O(V)\backslash O(V_{\mathbb{A}})} \Phi_L(h_{\mathbb{A}})\, \theta_\phi(g_{\mathbb{A}}, h_{\mathbb{A}})\, dh_{\mathbb{A}} && (4.1) \\ &= && (4.2) \end{aligned}$$

We first decompose $O(V_{\mathbb{A}})$ as a union of double cosets corresponding to the classes in the genus of $L$ (i.e. with respect to the adelic stabilizer $K_{\mathbb{A}}$), giving

$$O(V_{\mathbb{A}}) = \bigsqcup_{i\in I} O(V_F)\alpha_i K_{\mathbb{A}} \qquad \text{where } L_i := \alpha_i L$$

with $\mathrm{Gen}(L) = \bigsqcup_{i\in I}\{L_i\}$. This allows us to compute

$$\Theta(\Phi_L)(g_{\mathbb{A}}) = \int_{O(V_F)\backslash O(V_{\mathbb{A}})} \Phi_L(h_{\mathbb{A}})\,\theta_\phi(g_{\mathbb{A}}, h_{\mathbb{A}})\,dh_{\mathbb{A}} \tag{4.3}$$

$$= \int_{O(V_F)\backslash \bigsqcup_{i\in I} O(V_F)\alpha_i K_{\mathbb{A}}} \Phi_L(h_{\mathbb{A}})\,\theta_\phi(g_{\mathbb{A}}, h_{\mathbb{A}})\,dh_{\mathbb{A}} \tag{4.4}$$

$$= \sum_{i\in I} \int_{O(V_F)\backslash O(V_F)\alpha_i K_{\mathbb{A}}} \Phi_L(h_{\mathbb{A}})\,\theta_\phi(g_{\mathbb{A}}, h_{\mathbb{A}})\,dh_{\mathbb{A}} \tag{4.5}$$

$$= \sum_{i\in I} \int_{O(V_F)\backslash O(V_F)\alpha_i K_{\mathbb{A}}\alpha_i^{-1}} \Phi_L(h_{\mathbb{A}}\alpha_i)\,\theta_\phi(g_{\mathbb{A}}, h_{\mathbb{A}}\alpha_i)\,dh_{\mathbb{A}} \tag{4.6}$$

$$= \sum_{i\in I} \int_{O(V_F)\backslash O(V_F) K_{i,\mathbb{A}}} \Phi_L(h_{\mathbb{A}}\alpha_i)\,\theta_\phi(g_{\mathbb{A}}, h_{\mathbb{A}}\alpha_i)\,dh_{\mathbb{A}} \tag{4.7}$$

$$= \sum_{i\in I} \int_{(O(V_F)\cap K_{i,\mathbb{A}})\backslash K_{i,\mathbb{A}}} \Phi_L(h_{\mathbb{A}}\alpha_i)\,\theta_\phi(g_{\mathbb{A}}, h_{\mathbb{A}}\alpha_i)\,dh_{\mathbb{A}} \tag{4.8}$$

$$= \frac{1}{|\mathrm{Aut}(L_i)|} \sum_{i\in I} \int_{K_{i,\mathbb{A}}} \Phi_L(h_{\mathbb{A}}\alpha_i)\,\theta_\phi(g_{\mathbb{A}}, h_{\mathbb{A}}\alpha_i)\,dh_{\mathbb{A}} \tag{4.9}$$

$$\tag{4.10}$$

where the last step follows because $O(V_F) \cap K_{i,\mathbb{A}}$ is the finite group or rational automorphisms $\mathrm{Aut}(L_i)$, since $K_{i,\mathbb{A}}$ is the adelic stabilizer of $L_i$. We are also implicitly using the invariance of the Haar measure $dh_{\mathbb{A}}$ under right multiplication.

At this point we have unfolded our integral to the point where it factors as a product of local integrals, each of which we can try to evaluate separately. We first notice that for each summand we have an integral over $K_{i,\mathbb{A}} = \alpha_i K_{\mathbb{A}} \alpha_i^{-1}$, giving

$$\Theta(\Phi_L)(g_{\mathbb{A}}) = \frac{1}{|\mathrm{Aut}(L_i)|} \sum_{i\in I} \int_{K_{i,\mathbb{A}}} \Phi_L(h_{\mathbb{A}}\alpha_i)\,\theta_\phi(g_{\mathbb{A}}, h_{\mathbb{A}}\alpha_i)\,dh_{\mathbb{A}} \tag{4.11}$$

$$= \frac{1}{|\mathrm{Aut}(L_i)|} \sum_{i\in I} \int_{K_{\mathbb{A}}} \Phi_L(\alpha_i h_{\mathbb{A}})\,\theta_\phi(g_{\mathbb{A}}, \alpha_i h_{\mathbb{A}})\,dh_{\mathbb{A}} \tag{4.12}$$

whose integrals do not depend on $i \in I$. To analyze the internal integral, notice that $h_{\mathbb{A}} \in K_{\mathbb{A}}$, giving that

$$\Phi_L(\alpha_i h_{\mathbb{A}}) \neq 0 \iff \alpha_i h_{\mathbb{A}} \in O(V_F)\cdot 1 \cdot K_{\mathbb{A}} \tag{4.13}$$

$$\iff \alpha_i \in O(V_F)\cdot 1 \cdot K_{\mathbb{A}} \tag{4.14}$$

$$\iff \alpha_i = 1 \tag{4.15}$$

$$\tag{4.16}$$

and all terms with $\alpha_i \neq 1$ vanish. Thus

$$\Theta(\Phi_L)(g_{\mathbb{A}}) = \frac{1}{|\mathrm{Aut}(L_i)|} \int_{h_{\mathbb{A}} \in K_{\mathbb{A}}} \theta_\phi(g_{\mathbb{A}}, h_{\mathbb{A}})\, dh_{\mathbb{A}} \tag{4.17}$$

At this point we have to unwind the theta kernel to evaluate the integral, now heavily using the fact that this is a product of local integrals. Thus

$$\begin{aligned} \Theta(\Phi_L)(g_{\mathbb{A}}) &= \frac{1}{|\mathrm{Aut}(L_i)|} \int_{h_{\mathbb{A}} \in K_{\mathbb{A}}} \theta_\phi(g_{\mathbb{A}}, h_{\mathbb{A}})\, dh_{\mathbb{A}} && (4.18) \\ &= \frac{1}{|\mathrm{Aut}(L_i)|} \int_{h_{\mathbb{A}} \in K_{\mathbb{A}}} \sum_{\vec{v} \in V_F} (\mathcal{W}(g_{\mathbb{A}})\phi_{\mathbb{A}})(h_{\mathbb{A}}^{-1}\vec{v})\, dh_{\mathbb{A}} && (4.19) \\ & && (4.20) \end{aligned}$$

We now take advantage of the invariance of the theta lift under $\mathrm{Sp}(W_F)$ by adjusting the element $g_{\mathbb{A}}$ by an element of $g_F$ so that $g_{\mathbb{A}}$ has unipotent element $x \in \hat{\mathbb{Z}}$ and the Levi element $a \in \prod_p \mathbb{Z}_p^\times$. With these choices, we see that the local Weil representation element $\mathcal{W}(g_{\vec{v}})$ at each of the finite places acts trivially on the characteristic function $\phi_p(\vec{v})$ of $L_p$, so we are reduced to considering its action at the archimedean places $v \in \mathbf{a}$.

$$\begin{aligned} \Theta(\Phi_L)(g_{\mathbb{A}}) &= \frac{1}{|\mathrm{Aut}(L_i)|} \int_{h_{\mathbb{A}} \in K_{\mathbb{A}}} \sum_{\vec{v} \in V_F} (\mathcal{W}(g_{\mathbb{A}})\phi_{\mathbb{A}})(h_{\mathbb{A}}^{-1}\vec{v})\, dh_{\mathbb{A}} && (4.21) \\ &= \frac{1}{|\mathrm{Aut}(L_i)|} \int_{h_{\mathbb{A}} \in K_{\mathbb{A}}} \sum_{\vec{v} \in L} (\mathcal{W}(g_{\mathbf{a}})\phi_{\mathbf{a}})(h_{\mathbf{a}}^{-1}\vec{v})\, dh_{\mathbb{A}} && (4.22) \\ & && (4.23) \end{aligned}$$

Now we can write any element $g_{\mathbf{a}}$ as a product

$$g_{\mathbf{a}} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & 0 \\ 0 & {}^t\sqrt{y}^{-1} \end{bmatrix}$$

where $g_{\mathbf{a}} i = x + iy \in \mathcal{H}$. Then the action of the Weil representation in the sum gives the terms

$$
\begin{aligned}
(\mathcal{W}(g_{\mathbf{a}})\phi_{\mathbf{a}})(h_{\mathbf{a}}^{-1}\vec{v}) &= \left(\mathcal{W}\left(\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}\right)\mathcal{W}\left(\begin{bmatrix}\sqrt{y} & 0\\ 0 & {}^t\sqrt{y}^{-1}\end{bmatrix}\right)\phi_{\mathbf{a}}\right)(h_{\mathbf{a}}^{-1}\vec{v}) && (4.24)\\
&= y^{\frac{n}{2}}\left(\mathcal{W}\left(\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}\right)\phi_{\mathbf{a}}\right)(\sqrt{y}\,h_{\mathbf{a}}^{-1}\vec{v}) && (4.25)\\
&= y^{\frac{n}{2}}e^{2\pi\, ixQ(\vec{v})}\phi_{\mathbf{a}}(\sqrt{y}\,h_{\mathbf{a}}^{-1}\vec{v}) && (4.26)\\
&= y^{\frac{n}{2}}e^{2\pi\, ixQ(\vec{v})}e^{-2\pi Q(\sqrt{y}\,h_{\mathbf{a}}^{-1}\vec{v})} && (4.27)\\
&= y^{\frac{n}{2}}e^{2\pi\, ixQ(\vec{v})}e^{2\pi i\cdot iyQ(h_{\mathbf{a}}^{-1}\vec{v})} && (4.28)\\
&= y^{\frac{n}{2}}e^{2\pi\, ixQ(\vec{v})}e^{2\pi i\cdot iyQ(\vec{v})} && (4.29)\\
&= y^{\frac{n}{2}}e^{2\pi\, izQ(\vec{v})} && (4.30)\\
& && (4.31)
\end{aligned}
$$

Putting this back into the theta kernel, we have

$$
\begin{aligned}
\Theta(\Phi_L)(g_{\mathbb{A}}) &= \frac{1}{|\mathrm{Aut}(L_i)|}\int_{h_{\mathbb{A}}\in K_{\mathbb{A}}}\sum_{\vec{v}\in L}(\mathcal{W}(g_{\mathbf{a}})\phi_{\mathbf{a}})(h_{\mathbf{a}}^{-1}\vec{v})\,dh_{\mathbb{A}} && (4.32)\\
&= \frac{1}{|\mathrm{Aut}(L_i)|}\int_{h_{\mathbb{A}}\in K_{\mathbb{A}}}\sum_{\vec{v}\in L}y^{\frac{n}{2}}e^{2\pi\, izQ(\vec{v})}\,dh_{\mathbb{A}} && (4.33)\\
&= \frac{\mathrm{Vol}_{\mathbb{A}}(K)}{|\mathrm{Aut}(L_i)|}\sum_{\vec{v}\in L}y^{\frac{n}{2}}e^{2\pi\, izQ(\vec{v})} && (4.34)\\
&= \frac{1}{|\mathrm{Aut}(L_i)|}\sum_{\vec{v}\in L}y^{\frac{n}{2}}e^{2\pi\, izQ(\vec{v})} && (4.35)\\
& && (4.36)
\end{aligned}
$$

## 4.9 The Siegel-Weil Formula

The classical Siegel-Weil formula can also be viewed adelically in terms of the theta lifting, and this perspective allows one to extend its validity to quadratic forms in smaller numbers of variables where convergence becomes an issue. This was done in a series of papers by Kudla and Rallis, by using distributions to extend the region of convergence of both the (theta series and the Eisenstein series) to handle quadratic forms in ????? variables.

One key idea in the adelic treatment is to be able to explicitly identify the Eisenstein series in terms of the intertwining of the theta lift and an induced representation from the usual upper-triangular parabolic subgroup of $\mathrm{SL}_2$. This induced representation explains the appearance of an Eisenstein series in the Siegel-Weil formula, and various representation theoretic considerations (which we omit here) together with computation of the constant term for each allows us to identify this Eisenstein sereis explicitly.

We now sketch this intertwining or representations and the constant term computation to give a feeling for how the Siegel-Weil formula can be viewed from the perspective of theta liftings. For this we consider the theta lift of the constant function $F(h_{\mathbb{A}}) \equiv 1$ on $O(V_F)\backslash O(V_{\mathbb{A}})$, which from our previous computations corresponds to the classical weighted average

$$\frac{\sum_{Q_i \in \mathrm{Gen}(Q)} \frac{1}{|\mathrm{Aut}(Q_i)|} \theta_{Q_i}(z)}{\sum_{Q_i \in \mathrm{Gen}(Q)} \frac{1}{|\mathrm{Aut}(Q_i)|}}.$$

The classical Siegel-Weil formulas interpret this as an Eisenstein series, and our goal is to understand this fact adelically.

**Intertwining of the Theta Lift:** The intertwining

**Comparison of the constant terms:** The constant term of the theta lift is given (via the theory of Whittaker functionals) by the unipotent integral

$$\int_{N_F\backslash N_{\mathbb{A}}} \Theta(1)(g_{\mathbb{A}})\, dn_{\mathbb{A}} \tag{4.37}$$

$$= \int_{N_F\backslash N_{\mathbb{A}}} \int_{O(V_F)\backslash O(V_{\mathbb{A}})} \theta_{\phi}(n_{\mathbb{A}} g_{\mathbb{A}}, h_{\mathbb{A}})\, dh_{\mathbb{A}}\, dn_{\mathbb{A}} \tag{4.38}$$

$$= \int_{N_F\backslash N_{\mathbb{A}}} \int_{O(V_F)\backslash O(V_{\mathbb{A}})} \sum_{\vec{v} \in V_F} (\mathcal{W}(n_{\mathbb{A}} g_{\mathbb{A}})\phi_{\mathbb{A}})(h_{\mathbb{A}}^{-1}\vec{v})\, dh_{\mathbb{A}}\, dn_{\mathbb{A}} \tag{4.39}$$

$$= \int_{N_F\backslash N_{\mathbb{A}}} \int_{O(V_F)\backslash O(V_{\mathbb{A}})} \sum_{\vec{v} \in V_F} \psi_{\mathbb{A}}(n_{\mathbb{A}} Q(\vec{v}))(\mathcal{W}(g_{\mathbb{A}})\phi_{\mathbb{A}})(h_{\mathbb{A}}^{-1}\vec{v})\, dh_{\mathbb{A}}\, dn_{\mathbb{A}} \tag{4.40}$$

but the additive character $n_{\mathbb{A}} \mapsto \psi_{\mathbb{A}}(n_{\mathbb{A}} Q(\vec{v}))$ is non-trivial (meaning that the integral vanishes) whenever $Q(\vec{v}) \neq 0$, therefore the only have a contribution when $Q(\vec{v}) = 0$, or equivalently when $\vec{v} = \vec{0}$, giving constant term

$$\int_{N_F\backslash N_{\mathbb{A}}} \Theta(1)(g_{\mathbb{A}})\, dn_{\mathbb{A}} = (\mathcal{W}_{\psi}(g)\Phi)(0).$$

The constant term of the Eisenstein series is given by the term where $n_{\mathbb{A}} \in N_F$, and the other terms are seen to vanish due to an archimedean intertwining operator with another induced representation vanishing... so long as we are at a point $s_0$ far enough to the right.

## 4.10 The commutative diagram of modules relating weight $2$ and weight $3/2$ forms

In this section we describe an interesting diagram relating quadratic forms and modular forms in integral and half-integral weight.

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & M_2(p) \\
T \arrow[r] & M_{3/2}(4p) \arrow[u]
\end{tikzcd}$$

The conditions for the bottom vanishing are given in Gross's 1987 paper in terms of the vanishing of an $L$-function. The vanishing on the left is given by the local condition of vanishing, which can be interpreted as the Atkin-lehner involution on $f$, which is then interpereted in terms of the sign of the functional equation for $f$.

It is an interesting question to see how the choices of isomorphism on the top and right can be made compatible.

## 4.11 Computing other liftings

- general theta function for definite $Q$
    - spinor genus theta functions
    - discussion of the basis problem

# Chapter 5

# Other thoughts for discussion

## 5.1 Action of the class group of $\mathbb{Q}(\sqrt{-D})$ on the vectors of length $m$ in a ternary space.

This works by associating an order $A$ in a quaternion algebra, (whose trace zero dual lattice is $Q$)

Then any vector $\vec{v}$ of length $m$ gives a quadratic order $B$ in $A$ which is an order for $\mathbb{Q}(\sqrt{m})$. This embedded order now allows us to see any ideal class for $B$ as embedded in $A$. Then look at the right order of the left $A$-ideal generated by this order. But because $x$ is in the field, it is again in the right order. This $x$ gives another point on some other (possibly isomorphic) right order.

See Gross's 1987 paper, p134, for this.

## 5.2 Gross-Zagier formula and Gross's 1987 paper

## 5.3 Neighbors of lattices and the Eichler Commutation Relation

See Gonzalo's thesis, where he does this more generally for his $l$-theta series.

## 5.4 The Shimura lift in terms of its action on theta series

Paper of Ponomarev, and German book of Eichler. Also see the Eichler commutation relation in Gonzalo's thesis.

## 5.5 The correspondence between ternary forms and orders in a quaternion algebra

See Nipp paper. This corresondence is also used in the Brzezinski paper where he computes the type number 1 quaternion algebras!

## 5.6 His modular forms of wt 3/2 whose coefficients are modular forms of wt 3/2

# Part II

# Projects

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