Complex Dynamics and Unlikely Intersections

Arizona Winter School 2023 Problem Session

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1 Introduction

The aim of this problem set is to provide a self-contained walkthrough of the main results in Baker–DeMarco [BD11]. Given a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geq 2$ defined over $\mathbb{C}$, we say that a point $x \in \mathbb{P}^1(\mathbb{C})$ is *preperiodic* for f if the forward orbit $\{x, f(x), \dots f^n(x), \dots\}$ is finite. We will let $\mathrm{Prep}(f)$ denote the set of preperiodic points of f. The two main theorems of [BD11] are:

Theorem 1.1. *Let $d \geq 2$ be an integer and fix $a, b \in \mathbb{C}$. Then there exists infinitely many parameters $c \in \mathbb{C}$ such that both a and b are preperiodic for $z^d + c$ if and only if $a^d = b^d$.*

Theorem 1.2. *Let $f, g \in \mathbb{C}(z)$ be two rational functions of degree at least two. Then either $|\mathrm{Prep}(f) \cap \mathrm{Prep}(g)|$ is finite or $\mathrm{Prep}(f) = \mathrm{Prep}(g)$.*

Theorem 1.1 can be considered a dynamical analogue of the following result of Masser–Zannier [MZ10].

Theorem 1.3. *There are finitely many $\lambda \in \mathbb{C} \setminus \{0, 1\}$ such that both $P_\lambda = (2, \sqrt{2(2-\lambda)})$ and $Q_\lambda = (3, \sqrt{3(3-\lambda)})$ have finite order on the Legendre elliptic curve E_λ defined by $y^2 = x(x-1)(x-\lambda)$.*

In Theorem 1.3, we have a family of elliptic curves E_λ, along with two sections P_λ, Q_λ. For each of P, Q, there are infinitely many t's in which the specialization $\lambda \mapsto t$ produces a torsion point on the elliptic curve E_t. However, there are only finitely many t's where both sections specialize to a torsion point. Masser–Zannier results hold in greater generality for any two linearly independent sections P_λ, Q_λ.

Similarly in Theorem 1.1, we have a family of polynomial maps $z^d + c$, along with two sections given by the constant functions a and b. Then there are finitely many parameters c where both a and b are preperiodic points (the dynamical analog of torsion points), unless a and b are dynamically related.

The proofs of Theorems 1.1 and 1.2 rely on an adelic equidistribution theorem, along with properties of Julia and Mandelbrot sets. We will cover basic complex dynamics in Section 2 and then some potential theory in Section 3. In Section 4, we will develop a non-archimedean version of potential theory over the Berkovich projective line, and in Section 5 we will introduce and prove an adelic equidistribution theorem using both archimedean and non-archimedean potential theory. We will then deduce (simplified versions of) Theorems 1.1 and 1.2 in Section 6.

Version: Mar 2, 2023

Many of the problems are not exactly required to furnish a proof of Theorems 1.1 and 1.2 but are here to provide the reader additional background regarding the objects and tools that appear. If one wishes to have a "minimalist" approach to the main theorems, it is enough to attempt Problems $2, 4, 6, 7, 9, 10$ along with Sections 5 and 6.

2 Complex Dynamics

Complex dynamics is the study of iterations of rational functions over $\mathbb{C}$. We introduce the basic objects of study, namely the Fatou and Julia sets associated to the rational function, and develop the local theory of iteration near fixed points. We also give an overview of the Mandelbrot set.

The main reference for this section is [Mil11].

Problem 1. Montel's Theorem

We will assume the following theorem:

Theorem 2.1 (Riemann Uniformization)**.** *Any simply connected Riemann surface is isomorphic to either $\mathbb{P}^1$, $\mathbb{C}$ or the open disc $\mathbb{D}$.*

1. Let S be a Riemann surface (not necessarily compact) and $\pi : X \to S$ a topological covering space. Show that one can put a complex structure on X such that $\pi : X \to S$ is locally biholomorphic, and that all automorphisms of X as a covering space are holomorphic.

Thus any Riemann surface S has a universal cover that is also a Riemann surface. In particular, it is isomorphic to $\Gamma \backslash S'$ where S' is one of $\mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{D}$, and Γ is a discrete torsion-free subgroup of $\mathrm{Aut}(S')$.

2. Show that if S has $\mathbb{P}^1$ as an universal cover, then $S \simeq \mathbb{P}^1$.

3. Show that if S has $\mathbb{C}$ as an universal cover, then $S \simeq \mathbb{C}$, $\mathbb{C}^*$ or a complex torus $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subseteq \mathbb{C}$.

4. Consider the Poincaré metric

$$ds = \frac{2|dz|}{1 - |z|^2} \text{ for } z = x + iy \in \mathbb{D}$$

on the open disc. Let $\rho(x, y)$ be the distance corresponding to the Poincaré metric. Prove that

$$\rho(f(x), f(y)) \leq \rho(x, y)$$

for any holomorphic map $f : \mathbb{D} \to \mathbb{D}$.

If S has $\mathbb{D}$ as an universal cover, we say that S is *hyperbolic*. It naturally inherits a Poincaré metric from $\mathbb{D}$, which we will denote by ρ_S.

5. Let S, T be two hyperbolic surfaces and $f : S \to T$ a holomorphic map. Then

$$\rho_T(f(x), f(x')) \leq \rho_S(x, x').$$

From now on, we assume that S is hyperbolic. Let $\{f_n\}$ be a sequence of maps from S to an hyperbolic open $U \subset \mathbb{P}^1$.

6. Let $\{z_j\} \subset S$ be a countable dense subset. Show that one can choose a subsequence $\{g_m = f_{n_m}\}$ of $\{f_n\}$ such that $\lim_{m\to\infty} g_m(z_j) \in \overline{U}$ exists for all j.

7. Suppose that each of this limit points lies within U itself. Show that $\{g_n\}$ converges uniformly on any compact set K. (Hint: For a given compact set K and $\epsilon > 0$, choose finitely many z_j's such that $\rho_S(z, z_j) < \epsilon$ for all $z \in K$.)

8. Assume that one of the limit points a is on the boundary of U. Prove that $g_m(z)$ converges uniformly to a. Thus conclude the following theorem of Montel:

   Theorem 2.2 (Montel)**.** *Let S be a hyperbolic Riemann surface. If a collection $\mathcal{F}$ of holomorphic maps from S to $\mathbb{P}^1$ has three distinct points that never occur as values, then $\mathcal{F}$ is normal.*

Problem 2. Julia Sets of Rational Maps

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational function defined over $\mathbb{C}$ with degree $d \geq 2$. The *Fatou set* consists of points $z \in \mathbb{P}^1$ around which there exists an open neighborhood U such that $\{f, f^2, \ldots\}$ is a normal family when restricted to U. The *Julia set* $J(f)$ is the complement of the Fatou set.

1. Prove that $z \in J(f)$ if and only if $f(z) \in J(f)$.

2. Show that $J(f^n) = J(f)$.

3. Let z_1 be any point on the Julia set. Show that for any neighborhood U of z_1, the union of the forward images $f^n(U)$ omits at most two points of $\mathbb{P}^1$.

4. For $z_1 \in J(f)$, show that the set of iterated pre-images

$$\{z \mid f^n(z) = z_1 \text{ for some } n \geq 0\}$$

   is everywhere dense in $J(f)$.

For a periodic point $f^n(z) = z$ of minimal period n, the derivative

$$(f^n)'(z) = f'(z)f'(f(z)) \cdots f'(f^{n-1}(z))$$

is called the *multiplier* of the periodic orbit, denoted by λ. When $|\lambda| < 1$, $|\lambda| = 1$, or $|\lambda| > 1$, the cycle is called *attracting*, *indifferent*, or *repelling* respectively.

5. Show that repelling cycles lie in the Julia set.

Now let's further assume that $f(z)$ is a polynomial. We let $K_f = \{z \mid f^n(z) \not\to \infty\}$ be the *filled Julia set* of f.

6. Show that $\mathbb{P}^1 \setminus K_f$ is a connected neighborhood of ∞.

7. Show that the topological boundary ∂K_f is exactly the Julia set $J(f)$. (Hint: Use Montel's theorem.)

Problem 3. Linearization and Basins of Attraction

The following three problems explain the local behavior of a rational function f near periodic points. It suffices to study the case of fixed points, since a periodic point of f with period n is a fixed point of f^n.

Let z_0 be a fixed point of f with multiplier λ.

1. Prove that z_0 is an attracting fixed point (i.e., $|\lambda| < 1$) if and only if z_0 is *topologically attracting* in the following sense: there exists a neighborhood U of z_0 such that the sequence $\{f, f^2, \ldots\}$ converges uniformly on U to the constant function z_0.

2. Prove that z_0 is a repelling fixed point (i.e., $|\lambda| > 1$) if and only if z_0 is *topologically repelling* in the following sense: there exists a neighborhood U of z_0 such that for all $z \in U \setminus \{z_0\}$, there exists $n \geq 1$ such that $f^n(z) \notin U$.

From the Taylor series expansion

$$f(z) = z_0 + \lambda(z - z_0) + O((z - z_0)^2),$$

one expects that iterating f near z_0 "looks like" repeated multiplication by λ near 0. More precisely, we have the following:

Theorem 2.3 (Kœnigs linearization)**.** *Suppose $|\lambda| \neq 0, 1$. Then there exists a holomorphic change of coordinate $w = \phi(z)$ on a neighborhood U of z_0, such that $\phi(z_0) = 0$ and*

$$\phi(f(z)) = \lambda\phi(z)$$

for all $z \in U$. Moreover, ϕ is unique up to multiplication by a nonzero constant.

3. Prove the uniqueness statement in Theorem 2.3. (Hint: classify all power series which fix 0 and commute with $z \mapsto \lambda z$.)

4. Prove Theorem 2.3 for $0 < |\lambda| < 1$. (Hint: if $z_0 = 0$, consider $\lim_{n\to\infty} \lambda^{-n} f^n(z)$.)

5. Prove Theorem 2.3 for $|\lambda| > 1$. (Hint: consider f^{-1}, but note that this is not a rational function.)

Now suppose that z_0 is an attracting fixed point. The *basin of attraction* of z_0 is defined as

$$A(z_0) = \{z \mid \lim_{n\to\infty} f^n(z) = z_0\}.$$

The *immediate basin of attraction* of z_0 is the connected component of $A(z_0)$ containing z_0.

6. Prove that $A(z_0)$ is nonempty, open, and contained in the Fatou set of f.

7. Prove that $\partial A(z_0) = J(f)$.

8. Prove that the immediate basin of attraction of z_0 is also the component of the Fatou set of f containing z_0.

It turns out that the local linearization extends to a global linearization across the whole basin of attraction:

9. Prove that the Kœnigs coordinate ϕ extends to a holomorphic function on $A(z_0)$ such that

$$\phi(f(z)) = \lambda\phi(z)$$

for all $z \in A(z_0)$.

10. Prove that there is a maximal radius $0 < R < \infty$ such that ϕ^{-1} has an analytic continuation from a neighborhood of 0 to the disc $D(0, R)$. Deduce that the immediate basin of z_0 contains a critical point of f.

Problem 4. Superattracting Points and Böttcher Coordinates

Next we look at the case of $\lambda = 0$. In this case, we call z_0 a *superattracting* fixed point.

Note that we have the Taylor series expansion

$$f(z) = z_0 + a_m(z - z_0)^m + O((z - z_0)^{m+1}),$$

for some $m \geq 2$, $a_m \neq 0$. The integer m is called the *local degree* of f at z_0.

One might guess that iterating f near z_0 "looks like" repeated applications of the m-th power map near 0. More precisely, we have the following:

Theorem 2.4 (Böttcher)**.** *Suppose $\lambda = 0$. Then there exists a holomorphic change of coordinate $w = \phi(z)$ on a neighborhood U of z_0, such that $\phi(z_0) = 0$ and*

$$\phi(f(z)) = \phi(z)^m$$

for all $z \in U$. Moreover, ϕ is unique up to multiplication by a $(m - 1)$-th root of unity.

1. Prove the uniqueness statement in Böttcher's theorem. (Hint: classify all power series which fix 0 and commute with $z \mapsto z^m$.)

2. Prove the existence statement in Böttcher's theorem. (Hint: if $z_0 = 0$ and $a_m = 1$, consider $\lim_{n\to\infty}(f^n(z))^{1/m^n}$, where the m^n-th root is chosen with power series $z + \cdots$.)

Unlike the Kœnigs linearization, one does not expect to be able to globally extend the Böttcher coordinates to all of the basin of attraction, since the d-th root is multi-valued. However:

3. Prove that $\log|\phi|$ (where ϕ is the Böttcher coordinate) extends to a subharmonic function from $A(z_0)$ to $\mathbb{R} \cup \{-\infty\}$, harmonic except at the inverse iterates of z_0, such that

$$\log|\phi(f(z))| = m \log|\phi(z)|$$

for all $z \in A(z_0)$.

4. Prove that there is a maximal radius $0 < r \leq 1$ such that ϕ^{-1} has an analytic continuation from a neighborhood of 0 to the disc $D(0, r)$. Furthermore:

   (a) If $r = 1$, show that ϕ^{-1} is a conformal map from $\mathbb{D} = D(0, 1)$ to the immediate basin of z_0, and z_0 is the only critical point of f in the immediate basin.

(b) If $r < 1$, show that the immediate basin of z_0 contains another critical point of f, lying on the boundary of $\phi^{-1}(D(0,r))$.

Now consider the special case where f is polynomial, so that ∞ is a superattracting fixed point with local degree d.

5. Restate the results in this problem for a superattracting fixed point at ∞.

6. Suppose that all critical points of f have bounded orbit. Prove that the filled Julia set K_f and the Julia set $J(f) = \partial K_f$ are both connected, and that $\mathbb{C} \setminus K_f$ is conformally isomorphic to $\mathbb{C} \setminus \overline{\mathbb{D}}$.

Conversely, if at least one critical point of f has orbit escaping to ∞, then both K_f and $J(f)$ have uncountably many components; see [Mil11, Thm. 9.5] for details.

Problem 5. Parabolic Components

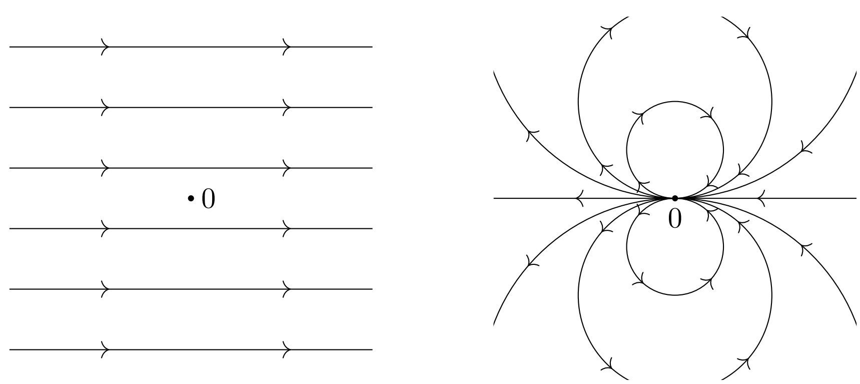
Finally we look at the case $|\lambda| = 1$. In this case, we call z_0 an *indifferent* or *neutral* fixed point. We will focus on the case where λ is in fact a root of unity, in which case we call z_0 a *parabolic* or *rationally neutral* fixed point. We will touch on the other case at the end of this problem.

1. Prove that every parabolic fixed point of f is in the Julia set $J(f)$. (Hint: Recall that $d = \deg f \geq 2$, so in particular no iterate of f is the identity; consider power series.)

The prototypical example of the behavior of iterating near a parabolic fixed point is given by the translation map $F(z) = z + 1$, with a parabolic fixed point at ∞, Equivalently, conjugating by $z \mapsto \frac{1}{z}$, we may consider the rational function

$$G(z) = \frac{1}{\frac{1}{z} + 1} = \frac{z}{z+1},$$

with a parabolic fixed point at 0. Notice that this map is attracting on one side of 0, and repelling on the other side, see figure.



$F(z) = z + 1$ $\qquad$ $G(z) = \frac{z}{z+1}$

Back to the general case. By replacing f with an iterate, let us assume that $\lambda = 1$. Also, by conjugating with a translation we may assume that $z_0 = 0$. Then we can write

$$f(z) = z + az^{n+1} + O(z^{n+2}),$$

where $n + 1 \geq 2$ is the *multiplicity* of the fixed point 0.

Just as in the above example, we expect that orbits near 0 get attracted along some directions, and repelled along other directions. If we write

$$f(z) = z(1 + az^n + O(z^{n+1})),$$

this suggests that the relevant directions are the *attraction vectors* $\mathbf{v}$ where $na\mathbf{v}^n = -1$, and *repulsion vectors* $\mathbf{v}$ where $na\mathbf{v}^n = +1$. Let us label these as $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{2n-1}$, where $\mathbf{v}_0$ is repelling and

$$\mathbf{v}_j = e^{\pi ij/n}\mathbf{v}_0,$$

so $\mathbf{v}_j$ is attracting (resp. repelling) when j is odd (resp. even).

Lemma 2.5. *If $\zeta_1 \mapsto \zeta_2 \mapsto \cdots$ is an orbit of f converging to 0 but not containing 0, then $\sqrt[n]{k}\zeta_k$ converges to some $\mathbf{v}_j$ with j odd.*

Similarly, if $\zeta'_1 \mapsto \zeta'_2 \mapsto \cdots$ is an orbit of f^{-1} converging to 0 but not containing 0, then $\sqrt[n]{k}\zeta'_k$ converges to some $\mathbf{v}_j$ with j even.

In other words, the asymptotic behavior of orbits which converge to 0 is $\zeta_k \sim k^{-1/n}\mathbf{v}_j$.

2. Let $\varphi(z) = c/z^n$, with $c = -1/(na)$. Check that $\varphi(\mathbf{v}_j) = (-1)^{j+1}$.

3. Let Δ_j be the open sector given by

$$\Delta_j = \left\{ re^{i\theta}\mathbf{v}_j \,\middle|\, r > 0,\ |\theta| < \frac{\pi}{n} \right\}.$$

Show that $\varphi(\Delta_j)$ is $\mathbb{C} \setminus \mathbb{R}_\pm$, the plane slit along either the positive or the negative real axis (depending on j), and that φ is a conformal map from Δ_j to its image.

4. Let $\psi_j : \mathbb{C} \setminus \mathbb{R}_\pm \to \Delta_j$ be the corresponding branch of φ^{-1}, and write $F_j = \varphi \circ f \circ \psi_j$. Verify that as $|w| \to \infty$, we have

$$F_j(w) = w + 1 + O(|w|^{-1/n}).$$

5. Show that there exists some large $R > 0$ such that

$$|w| > R \implies \operatorname{Re} F_j(w) > \operatorname{Re} w + \frac{1}{2}.$$

6. Let $w_k = \varphi(\zeta_k)$. Show that $\frac{w_k}{k} \to 1$ as $k \to \infty$. (Hint: show that $w_{k+1} - w_k \to 1$.) Now finish the proof of Lemma 2.5.

Each attraction vector corresponds to a *parabolic basin of attraction* $\mathcal{A}_j = \mathcal{A}(z_0, \mathbf{v}_{2j-1})$, the set of points ζ_1 for which the orbit $\zeta_1 \mapsto \zeta_2 \mapsto \cdots$ converges to z_0 from the direction $\mathbf{v}_{2j-1}$. The *immediate basin of attraction* $\mathcal{A}_j^0$ is the connected component of $\mathcal{A}_j$ which contains ζ_k for all large k for any such orbit.

7. Prove that each parabolic basin $\mathcal{A}_j$ is contained in the Fatou set $\mathbb{P}^1 \setminus J(f)$, while its boundary $\partial\mathcal{A}_j$ is contained in the Julia set $J(f)$.

Suppose that f is univalent on some neighborhood N of z_0. An open set $\mathcal{P} \subseteq N$ is called an *attracting petal* for f for the attracting vector $\mathbf{v}_j$ if

- f maps $\mathcal{P}$ into itself; and
- an orbit $\zeta_1 \mapsto \zeta_2 \mapsto \cdots$ converges to z_0 from the direction $\mathbf{v}_j$ if and only if $\mathcal{P}$ contains ζ_k for all large k.

Similarly, if $f : N \to N'$ is a conformal map, then an open set $\mathcal{P} \subseteq N'$ if called a *repelling petal* for f for the repelling vector $\mathbf{v}_j$ if it is an attracting petal for $f^{-1} : N' \to N$ for $\mathbf{v}_j$.

8. With notation as in the proof of the lemma above, define $\mathcal{P}_j = \psi_j(\Omega_R)$, where $\Omega_R = \{u + iv \mid u + |v| > 2R\}$. Show that $\mathcal{P}_j$ is an attracting petal for f for the attracting vector $\mathbf{v}_j$.

Hence we have the following result, due to Leau, Julia and Fatou:

Theorem 2.6 (Parabolic flower theorem). *Let z_0 be a fixed point of f of multiplicity $n+1 \geq 2$. Then in any neighborhood of z_0 there exists $\mathcal{P}_j$, with indices taken modulo $2n$, such that:*

- *$\mathcal{P}_j$ is simply connected;*
- *$\mathcal{P}_j$ is an attracting (resp. repelling) petal for j odd (resp. even); and*
- *$\{z_0\} \cup \mathcal{P}_0 \cup \cdots \cup \mathcal{P}_{2n-1}$ is a neighborhood of z_0.*

Moreover, if $n \geq 2$, the set $\mathcal{P}_j \cap \mathcal{P}_{j+1}$ is simply connected and disjoint from the other $\mathcal{P}_k$; if $n = 1$, $\mathcal{P}_0 \cap \mathcal{P}_1$ has two simply connected components.

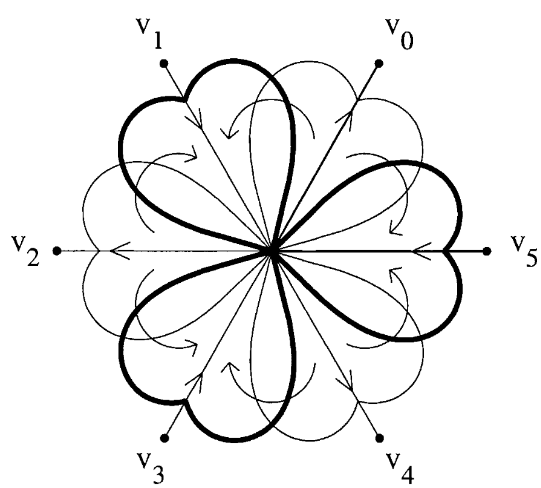


Figure 1: Flower with three attracting petals (in bold) and three repelling petals [Mil11, Fig. 22].

There is a corresponding linearization theorem for parabolic fixed points, stating that the dynamics in any petal is conjugate to the translation $z \mapsto z + 1$.

Theorem 2.7. *For any attracting or repelling petal $\mathcal{P}$, there is a conformal map $\alpha : \mathcal{P} \to \mathbb{C}$, unique up to translation, such that*

$$\alpha(f(z)) = \alpha(z) + 1$$

for all $z \in \mathcal{P} \cap f^{-1}(\mathcal{P})$.

9. With notation as above, fix $\widehat{w} \in \Omega_R$. Prove that the sequence of functions

$$\beta_k(w) = F_j^k(w) - F_j^k(\widehat{w})$$

converges locally uniformly to a conformal map β on Ω_R, satisfying

$$\beta(F(w)) = \beta(w) + 1.$$

10. Finish the proof of Theorem 2.7.

The above discussion generalizes easily to parabolic fixed points with multiplier $\lambda = e^{2\pi i p/q}$, but we can say more about the multiplicity in this case.

11. Prove that if the multiplier λ is a primitive q-th root of unity, and z_0 is a fixed point of f^q of multiplicity $n+1$, then n is a multiple of q. (Hint: consider the attraction vectors corresponding to $\zeta_1 \mapsto \zeta_{q+1} \mapsto \cdots$ and $\zeta_2 \mapsto \zeta_{q+2} \mapsto \cdots$.)

We finish with some remarks on *irrationally indifferent* or *irrationally neutral* fixed points, namely those with $|\lambda| = 1$ and λ not a root of unity. The central problem in this case is whether f can be locally linearized, i.e., if there exists a local holomorphic change of coordinate $z = h(w)$ such that $z_0 = h(0)$ and

$$f(h(w)) = h(\lambda w)$$

for w near 0.

12. Prove that if f is locally linearizable around an indifferent fixed point z_0, then z_0 is in the Fatou set $\mathbb{P}^1 \setminus J(f)$. (The converse also holds, see [Mil11, Lem. 11.1].)

It turns out that the issue of local linearizability is quite subtle, and depends on how well $\frac{1}{2\pi} \arg \lambda$ can be approximated by rational numbers.

Theorem 2.8 (Cremer)**.** *If $\inf_q |\lambda^q - 1|^{1/d^q} = 0$, then f is not locally linearizable around z_0.*

Theorem 2.9 (Siegel)**.** *If there exists $C, N > 0$ such that $|\lambda^q - 1| \geq Cq^{-N}$ for all $q \geq 1$, then f is locally linearizable around z_0.*

We refer the interested reader to [Mil11, Ch. 11; CG93, Ch. II.6] for further results.

Problem 6. The Mandelbrot Set

The main reference for this problem is [CG93, §VIII].

The simplest non-trivial family of rational functions is arguably the family of quadratic polynomials. We first note that this is essentially a one-parameter family:

1. Prove that every quadratic polynomial is conjugate to a unique polynomial of the form $P_c(z) \coloneqq z^2 + c$.

The Julia sets $J(P_c)$ are usually fractals and cannot be described simply, with two exceptions.

2. Compute $J(P_0)$.

3. Prove that $J(P_{-2}) = [-2, 2]$. (Hint: Check that $P_{-2}(2\cos\theta) = 2\cos 2\theta$.)

A basic object associated with the family of quadratic polynomials is the *Mandelbrot set*

$$\mathcal{M} \coloneqq \{c \in \mathbb{C} \mid J_{P_c} \text{ is connected}\}.$$

Since P_c has exactly one critical point (namely 0), by the last result in Problem 4 we have equivalently

$$\begin{aligned}\mathcal{M} &= \{c \in \mathbb{C} \mid (P_c^n(0))_{n\geq 1} \text{ is bounded}\}\\ &= \{c \in \mathbb{C} \mid 0 \in K_{P_c}\}.\end{aligned}$$

4. Prove that $\mathcal{M} \subseteq D(0, 2)$.

5. Prove that for all $c \in \mathcal{M}$, we have $K_{P_c} \subseteq D(0, 2)$. In particular, we have $|P_c^n(0)| \leq 2$ for all $n \geq 1$.

6. Deduce that $\mathcal{M}$ is closed, and every component of $\mathcal{M}$ is simply connected. (Hint: Show by the maximum modulus principle that $\mathbb{C} \setminus \mathcal{M}$ has no bounded components.)

7. Prove that $\mathcal{M} \cap \mathbb{R} = [-2, \frac{1}{4}]$.

We now identify the largest components of $\mathcal{M}$. By the last result in Problem 3, every basin of attraction for P_c attracts some critical point; hence if P_c has an attracting cycle, then $c \in \mathcal{M}$.

8. Characterize all c such that P_c has an attracting fixed point. Deduce that $\mathcal{M}$ contains the cardioid
$$C = \left\{ \frac{\lambda}{2} - \frac{\lambda^2}{4} \;\middle|\; |\lambda| < 1 \right\}.$$

9. Characterize all c such that P_c has an attracting cycle of period 2. Deduce that $\mathcal{M}$ contains the disc $D(-1, \frac{1}{4})$.

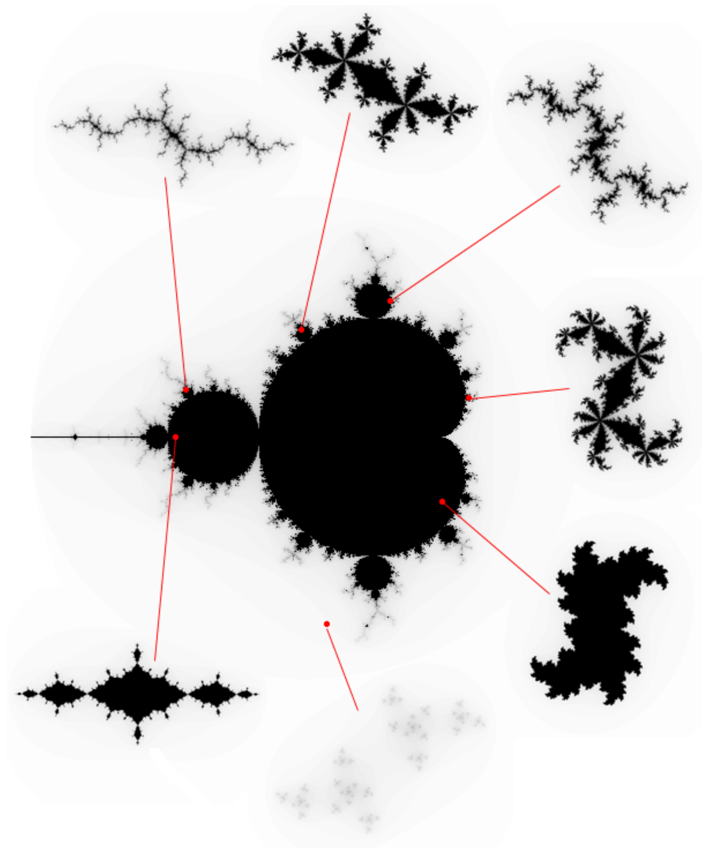


Figure 2: The Mandelbrot set $\mathcal{M}$ (center), with K_{P_c} for selected values of c [Bou01].

3 Potential Theory

We start off with some basic properties of subharmonic functions and then show how potential theory allows one to prove equidistribution statements. We then introduce the equilibrium measure of a compact set K and end off with another application of potential theory to the Mandelbrot set $\mathcal{M}$.

The main reference for this section is [Ran95].

Problem 7. Subharmonic Functions

Let X be a topological space. A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is called *upper semicontinuous* if $u^{-1}((-\infty, \alpha))$ is open for all $\alpha \in \mathbb{R}$.

1. Let u be an upper semicontinuous function on a compact set K. Show that u attains a maximum on K.

2. Let u be an upper semicontinuous function on a metric space (X, d). Define

$$\phi_n(x) = \sup_{y \in X}(u(y) - nd(x, y)).$$

Show that each $\phi_n : X \to \mathbb{R}$ is continuous and $\phi_1 \geq \phi_2 \geq \cdots \geq u$ with $\lim_{n\to\infty} \phi_n = u$.

Let U be an open subset of $\mathbb{C}$. A function $u : U \to \mathbb{R} \cup \{-\infty\}$ is said to be *subharmonic* if it is upper semicontinuous and for any $w \in U$, there exists $\rho > 0$ such that

$$u(w) \leq \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{it})dt$$

for any $0 \leq r < \rho$.

3. Let f be a holomorphic function on an open set U. Show that $\log|f|$ is subharmonic.
4. Let $u \in C^2(U)$ be twice continuously differentiable. Show that u is subharmonic if and only if $\triangle u \geq 0$.

We now introduce the notion of smoothing. Let $u : U \to \mathbb{R} \cup \{-\infty\}$ be a locally integrable function and let $\phi : \mathbb{C} \to \mathbb{R}$ be a continuous function with support in $D(0, r)$. The convolution $u * \phi$ is defined as

$$(u * \phi)(z) = \int_{\mathbb{C}} u(z - w)\phi(w)dA(w).$$

5. Show that for any subharmonic function $u : U \to \mathbb{C}$, there exists a sequence of smooth subharmonic functions (u_n) such that $u_1 \geq u_2 \cdots \geq u$ and $\lim_{n\to\infty} u_n(z) = u$ for all $z \in U$. (Remark: the functions u_n might be defined on a smaller open subset $U_n \subseteq U$.
6. Let u be a subharmonic function on U. Show that $\triangle u \geq 0$ in the sense of distributions.

Let E be a subset of $\mathbb{C}$. If $E \subseteq \{u(z) = -\infty\}$ for some non-constant subharmonic function u, we say that E is a *small* set.

7. Let u be a subharmonic function on U. Show that if $\limsup_{z\to\zeta} u(z) \leq 0$ for all $\zeta \in \partial U$, then $u \leq 0$ on U.
8. Let U be an open subset of $\mathbb{C}$, let E be a closed small set and let u be a subharmonic function on $U \setminus E$. Suppose that u is locally bounded on U. Prove that u has an unique subharmonic extension to the whole of U. (Hint: extend u by $u(w) = \limsup_{z\to w, z\in U\setminus E} u(z)$. To show it is subharmonic, consider $u + \epsilon v$ where $E = \{v(z) = -\infty\}$.)
9. Deduce a generalization of Riemann's removable singularity theorem: let $E \subseteq \mathbb{C}$ be a countable closed subset (not necessarily discrete) and let f be a holomorphic function on $\mathbb{C} \setminus E$ that is locally bounded on $\mathbb{C}$. Show that f extends to an entire function.

Problem 8. Brolin's Theorem

1. Let T be a distribution on $\mathbb{R}$ that is positive, i.e. $T(f) \geq 0$ for all $f \geq 0$. Show that T is of order zero, i.e. it extends to a continuous linear functional on the space of all compactly supported continuous functions.
2. Let (u_n) be a sequence of uniformly bounded subharmonic functions on $U \subseteq \mathbb{R}^2$. Show that if u_n converges uniformly to u, then $\triangle u_n$ converges weakly to $\triangle u$ too.

Now let's fix a polynomial $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geq 2$.

3. Let $G_n(z) = \frac{1}{d^n} \log^+ |f^n(z)|$. Show that G_n converges uniformly to some function G_f. How does G_f relate to the Böttcher coordinates of f from Problem 4?

4. Show that on $\mathbb{C} \setminus K(f)$, the functions G_n are all harmonic. Conclude that $\lim_{n\to\infty} G_n = \triangle G_f$ and that $\triangle G_f$ is supported on the Julia set $J(f)$.

5. Now let $a \in \mathbb{C}$ and consider the measure δ_a. Convolving with a disc of radius ϵ gives us a measure $(\delta_a)_\epsilon$ with a smooth potential $V_\epsilon(z)$. Show there exists a constant $C > 0$ such that
$$|V_\epsilon(z) - \log^+ |z|| \leq C.$$

6. Show that
$$\lim_{n\to\infty} \frac{1}{d^n} V_\epsilon(f^n(z)) = G_f(z)$$
and hence conclude that $(f^n)^*(\delta_{a,\epsilon}) = \mu_f$ where $\mu_f = \triangle G_f$.

Now assume the following lemma:

Lemma 3.1. *For any $\delta > 0$, there exists n_0 and k_0 so that at least $(1-\delta)d^n$ elements of $f^{-n}(z_0)$ lie in a connected component of $f^{-n}(U)$ having diameter $\leq \delta$, for all $k \geq k_0$ and $n \geq n_0$ and any sufficiently small neighborhood U of z_0.*

7. Using the lemma, prove that
$$\frac{1}{d^n} \sum_{f^n(z)=a} \delta_z \to \mu_f.$$

Let's try to prove the lemma under the assumption that $\{f^n(z) = a\}$ are all distinct and we can find a neighborhood $a \in U$ such that $f^{-n}(U)$ contain no critical points. We will need the following distortion theorem.

Theorem 3.2 (Koebe's distortion theorem)**.** *Let $f : \mathbb{D} \to \mathbb{C}$ be a univalent function such that $f(0) = 0$ and $f'(0) = 1$. Then*
$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

8. Show that each z satisfying $f^n(z) = a$, is contained in an unique connected component of $f^{-n}(U)$, denoted by U_z, and $f^{-n} : U \to U_z$ is a biholomorphism.

9. Now apply the Koebe's distortion theorem to conclude that for a suitable choice of U, each U_z is almost a round disc, i.e. its diameter is bounded by a fixed constant times the area.

10. Conclude Lemma 3.1. (Hint: The total area is bounded by some constant.)

Problem 9. The Equilibrium Measure

Let μ be a finite Borel measure on $\mathbb{C}$ with compact support. Its *potential function* is the function $p_\mu : \mathbb{C} \to \mathbb{R} \cup \{-\infty\}$ defined by

$$p_\mu(z) = \int \log|z-w| d\mu(w).$$

1. Show that $p_\mu(z)$ is subharmonic on $\mathbb{C}$, harmonic on $\mathbb{C} \setminus (\operatorname{supp}\mu)$, and
$$p_\mu(z) = \mu(\mathbb{C}) \log|z| + O(|z|^{-1}).$$

Keeping the same assumptions on μ, its *energy* $I(\mu)$ is defined as

$$I(\mu) = \int\int \log|z-w| d\mu(z) d\mu(w) = \int p_\mu(w) d\mu(w).$$

A set $E \subset \mathbb{C}$ is called *polar* if $I(\mu) = -\infty$ for every finite Borel measure $\mu \neq 0$ for which $\operatorname{supp}\mu$ is a compact subset of E. Polar sets are the "negligible" sets in potential theory.

2. Let μ be a finite Borel measure with compact support such that $I(\mu) > -\infty$. Show that $\mu(E) = 0$ for every Borel polar set E.

3. Show that every Borel polar set has Lebesegue measure zero.

4. Show that a countable union of Borel polar sets is polar.

Given a compact set K, let $\mathcal{P}(K)$ be the space of all Borel probability measures on K. If $\nu \in \mathcal{P}(K)$ maximizes the energy $I(\nu)$, we say that ν is an *equilibrium measure* for K and that the *logarithmic capacity* of K is $c(K) = e^{I(\nu)}$.

5. Let (μ_n) be a sequence of measures supported on K that converges weakly to a measure μ. Show that $\limsup_{n\to\infty} I(\mu_n) \leq I(\mu)$. Conclude that every compact set in $\mathbb{C}$ has an equilibrium measure.

We now aim to show Frostman's theorem.

Theorem 3.3 (Frostman)**.** *Let K be a compact set and ν an equilibrium measure for K. Then $\rho_\nu \geq I(\nu)$ on $\mathbb{C}$ and $\rho_\nu = I(\nu)$ on $K \setminus E$ for some polar subset E.*

Heuristically, one expects the potential of points to be all equal, else one could let the charge flow from a lower to a higher point to obtain a higher amount of energy.

6. Let
$$K_n = \left\{ z \in K \,\middle|\, \rho_\nu(z) \geq I(\nu) + \frac{1}{n} \right\}$$
and
$$L_n = \left\{ z \in \operatorname{supp}\nu \,\middle|\, \rho_\nu(z) < I(\nu) - \frac{1}{n} \right\}.$$
Show it suffices to show that K_n is polar and L_n is empty for each $n \geq 1$.

7. Suppose some K_n is non-polar. Show there exists some disc $D(z_0, r)$ with $z_0 \in \operatorname{supp} \nu$ such that $\rho_\nu < I(\nu) + \frac{1}{2n}$ on $D(z_0, r)$.
8. Let $a = \nu(D(z_0, r))$. Define a signed measure σ by $\sigma = \mu$ on K_n, and $\sigma = -\nu/a$ on $D(z_0, r)$ and 0 everywhere else. Let $\nu_t = \nu + t\sigma$. Show that $I(\nu_t) > I(\nu)$ for t sufficiently small, which is a contradiction.
9. Now suppose that L_n is non-empty for some n. Show there exists a disc $D(z_1, s)$ such that $\rho_\nu < I(\nu) - \frac{1}{n}$ on $D(z_1, s)$. Using the fact that K_n is polar for all n, obtain a contradiction.

Using this, we can show that the equilibrium measure is unique.

10. Let μ_1 and μ_2 be two equilibrium measures of K. Show that $\rho_{\mu_1} = \rho_{\mu_2}$ everywhere on $\mathbb{C}$ and conclude that $\mu_1 = \mu_2$.

We can also give another characterization of the equilibrium measure using Green functions.

11. Let K be a compact set with $c(K) > 0$ and let Ω_K be the unbounded connected component of $\mathbb{C} \setminus K$. Show there exists an unique function $g_K(\cdot, \infty) : \Omega_K \to \mathbb{R}$ such that
    (a) g_K is harmonic on Ω_K and bounded on all compact subsets;
    (b) $g_K(z) - \log |z|$ is bounded on a neighborhood of ∞;
    (c) $\lim_{z \to w} g_K(z) = 0$ for all $w \in \partial\Omega_K$ outside of a polar set.
12. Show that in fact we have $\lim_{z\to\infty}(g_K(z) - \log |z|) = -\log c(K)$.

We call $g_K(\cdot, \infty)$ the *Green's function* of K with respect to ∞.

13. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a polynomial of degree $d \geq 2$ and let $G_f(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)|$. Show that $\frac{1}{2\pi}\triangle G_f$ is the equilibrium measure for the Julia set $J(f)$ of f.

Problem 10. Connectedness of $\mathcal{M}$

The main reference for this problem is [CG93, §VIII].

As in Problem 6, let $P_c(z) = z^2 + c$ and $\mathcal{M}$ be the Mandelbrot set. In this problem, we will explicitly construct the uniformizing map $\mathbb{P}^1 \setminus \mathcal{M} \to \mathbb{P}^1 \setminus \overline{\mathbb{D}}$, and hence prove:

Theorem 3.4 (Douady–Hubbard, Sibony)**.** *$\mathcal{M}$ is connected.*

Recall from Problem 4 that there are Böttcher coordinates $\phi_c(z)$ for P_c around ∞, given by

$$\phi_c(z) = \lim_{n\to\infty} (P_c^n(z))^{1/2^n},$$

such that $\phi_c(P_c(z)) = \phi_c(z)^2$. Furthermore, there exists a harmonic extension of $\log |\phi_c|$ to $\mathbb{C} \setminus K_{P_c}$, which is equal to G_{P_c} as defined in Problem 8. Since 0 is the only critical point of P_c, by another result from Problem 4 we see that ϕ_c is well-defined and analytic in the region $\{z \in \mathbb{C} \mid G_{P_c}(z) > G_{P_c}(0)\}$.

1. Check that $\phi_c(z) = z + o(1)$ near ∞. Deduce that the logarithmic capacity of K_{P_c} is 1.

2. Prove that we have the product representation

$$\phi_c(z) = z \prod_{n=0}^{\infty} \left(1 + \frac{c}{(P_c^n(z))^2}\right)^{1/2^{n+1}},$$

choosing the appropriate branch of the root.

Our next goal is to show the following regularity result for G_{P_c}.

Proposition 3.5. *Let $R > 10$. Then G_{P_c} is uniformly Hölder continuous for $|c| \leq R$; in other words, there exists $\alpha_R, C_R > 0$ such that*

$$|G_{P_c}(z_1) - G_{P_c}(z_2)| \leq C_R |z_1 - z_2|^{\alpha_R}$$

for all $|c| \leq R$ and $z_1, z_2 \in \mathbb{C}$.

3. Prove that $G_{P_c}(z)$ is uniformly bounded above on $|c| \leq R$, $|z| \leq R$. (Hint: use the product representation of $\phi_c(z)$.)

4. Take $|c| \leq R$ and $|z| \leq R$ such that $z \notin K_{P_c}$, and let N be minimal such that $|P_c^N(z)| > R$. Prove that
$$\operatorname{dist}(P_c^N(z), K_{P_c}) \leq (2R)^N \operatorname{dist}(z, K_{P_c}).$$
(Hint: P_c has Lipschitz constant $2R$ on $\overline{D}(0, R)$; why?)

5. Deduce that there exists $\alpha_R, C'_R > 0$ such that

$$G_{P_c}(z) \leq C'_R \operatorname{dist}(z, K_{P_c})^{\alpha_R}$$

for all $|c| \leq R$ and $|z| \leq R$.

6. Using the following inequality, prove Proposition 3.5.

Proposition 3.6 (Harnack's inequality)**.** *Let $f \geq 0$ be a non-negative function, continuous on the closed ball $\overline{D}(x_0, r)$ in the plane, and harmonic in its interior. Then for all $|x - x_0| = \rho < r$, we have*

$$\frac{r - \rho}{r + \rho} f(x_0) \leq f(x) \leq \frac{r + \rho}{r - \rho} f(x_0).$$

7. Deduce that the family $\{G_{P_c} \mid |c| \leq R\}$ is equicontinuous on compact sets.

This gives us the following continuity result:

Theorem 3.7. *If $c_n \to c$, then $G_{P_{c_n}} \to G_{P_c}$ uniformly on $\mathbb{C}$.*

8. Suppose some subsequence of $G_{P_{c_n}}$ converges locally uniformly to H. Prove that H is continuous on $\mathbb{C}$, and harmonic on $\Omega = \{z \mid H(z) > 0\}$.

9. Prove that Ω is connected and unbounded. (Hint: use the maximum principle.)

10. Prove Theorem 3.7. (Hint: show that $H = G_{P_c}$ on Ω, and that $\Omega = \mathbb{C} \setminus K_{P_c}$.)

For $c \in \mathbb{C} \setminus \mathcal{M}$, we have $G_{P_c}(0) > 0$, so

$$G_{P_c}(c) = 2G_{P_c}(0) > G_{P_c}(0).$$

Hence we may define the function Φ on $\mathbb{C} \setminus \mathcal{M}$ by

$$\Phi(c) := \phi_c(c) = c \prod_{n=0}^{\infty} \left(1 + \frac{c}{(P_c^n(c))^2}\right)^{1/2^{n+1}}.$$

We now show that Φ is the desired conformal isomorphism $\mathbb{C} \setminus \mathcal{M} \to \mathbb{C} \setminus \overline{\mathbb{D}}$.

11. Prove that Φ is analytic, has a simple pole at ∞, and

$$\log |\Phi(c)| = G_{P_c}(c) = 2G_{P_c}(0).$$

12. Prove that for $c \in \mathbb{C} \setminus \mathcal{M}$, we have $|\Phi(c)| > 1$, and $|\Phi(c)| \to 1$ as c approaches $\mathcal{M}$. (Hint: Theorem 3.7 implies $G_{P_c}(c) \to 0$.)

13. Complete the proof of Theorem 3.4. (Hint: use the argument principle.)

14. Check that $\Phi(c) = c + o(1)$ near ∞. Deduce that the logarithmic capacity of $\mathcal{M}$ is 1.

One of the biggest open problems in complex dynamics is the MLC conjecture, which states that $\mathcal{M}$ is locally connected. This has important implications for the dynamics of the quadratic family, such as density of hyperbolicity; see [Ben17] for a recent survey. In our language, MLC is equivalent (by a result of Carathéodory) to the claim that Φ^{-1} has a continuous extension to $\partial\mathbb{D}$.

4 The Berkovich Projective Line

Over a non-archimedean field $\mathbb{C}_v$, a good theory of dynamics can be developed if one uses the Berkovich space $\mathbb{P}^1_{\mathrm{Berk},v}$ instead of the usual $\mathbb{P}^1(\mathbb{C}_v)$. Benedetto's text [Ben19] is a good introduction to the dynamical theory, and if one wishes to develop a non-archimedean potential theory, Baker and Rumely's text [BR10] is a good reference.

Let $\mathbb{C}_v$ denote a complete, algebraically closed field equipped with a non-trivial non-archimedean absolute value $|\cdot|_v$. For a ring A, a *multiplicative seminorm* is a function $|\cdot| : A \to [0, \infty)$ such that

1. $|0| = 0$ and $|1| = 1$;
2. for all $f, g \in A$, $|fg| = |f| \cdot |g|$;
3. for all $f, g \in A$, $|f + g| \leq |f| + |g|$.

The *Berkovich affine line*, denoted by $\mathbb{A}^1_{\mathrm{Berk},v}$, is the set of all multiplicative seminorms $\zeta = |\cdot|_\zeta$ that extends the norm on $\mathbb{C}_v$. We give $\mathbb{A}^1_{\mathrm{Berk},v}$ the *Gelfand topology*, which is the weakest topology such that $f \mapsto |f|_\zeta$ is a continuous map on $\mathbb{A}^1_{\mathrm{Berk},v}$ for every $f \in \mathbb{C}_v[z]$.

Problem 11. Points of $\mathbb{A}^1_{\mathrm{Berk},v}$

1. Let $a \in \mathbb{C}_v$. Show that $f \mapsto |f(a)|_v$ defines a multiplicative seminorm on $\mathbb{C}_v[z]$.

   These are the *type I points* or the *classical points*, and can be identified with $\mathbb{C}_v$.
   Let $D(a, r) \subseteq \mathbb{C}_v$ be the disc centered at a with radius r. For any $f \in \mathbb{C}_v[z]$, let

$$|f|_{D(a,r)} = \sup_{z \in D(a,r)} |f(z)|_v.$$

2. If $f(z) = \sum_{n \geq 0} c_n (z - a)^n$, show that $|f|_{D(a,r)} = \sup_{n \geq 0}\{|c_n| r^n\}$.
3. Prove that $|\cdot|_{D(a,r)}$ defines a multiplicative seminorm.

We will write $\zeta(a, r)$ for $|\cdot|_{D(a,r)}$ when we view it as a point in $\mathbb{A}^1_{\mathrm{Berk},v}$. If $r \in |\mathbb{C}_v^\times|$, this defines for us a *type II point*. If $r \notin |\mathbb{C}_v^\times|$, this defines for us a *type III point*.

4. Let $D_1 \supseteq D_2 \supseteq \cdots$ be a nested sequence of closed discs such that $\bigcap_{n=1}^\infty D_n = \emptyset$, but the radii r_n do not converge to zero. Let $\zeta_n \in \mathbb{A}^1_{\mathrm{Berk},v}$ be the multiplicative seminorm corresponding to D_n. Show that $|f|_\zeta = \lim_{n \to \infty} |f|_{\zeta_n}$ defines a multiplicative seminorm on $\mathbb{C}_v[z]$. This defines for us a *type IV point*.

It is a theorem of Berkovich that any point on $\mathbb{A}^1_{\mathrm{Berk},v}$ is one of the four types above. Type IV points occur only when the field $\mathbb{C}_v$ is not spherically complete.

Here's a proof sketch to see that $\mathbb{A}^1_{\mathrm{Berk},v}$ is path connected. Given two points $x, y \in \mathbb{C}_v$, we let $r = |x - y|$ and so $\overline{D}(x, r) = \overline{D}(y, r)$. Viewing x as $D(x, 0)$, we increase the radius of our disc to r to get a path from x to $\zeta(x, r)$. Then now viewing it as $\zeta(y, r)$, we decrease the radius down to 0 to get to y. This gives us a path from x to y.

Let

$$\overline{D}_{an}(a,r) = \{\zeta \in \mathbb{A}^1_{\mathrm{Berk},v} \mid |z-a|_\zeta \le r\}$$

be the *closed Berkovich disc* centered at a of radius r and

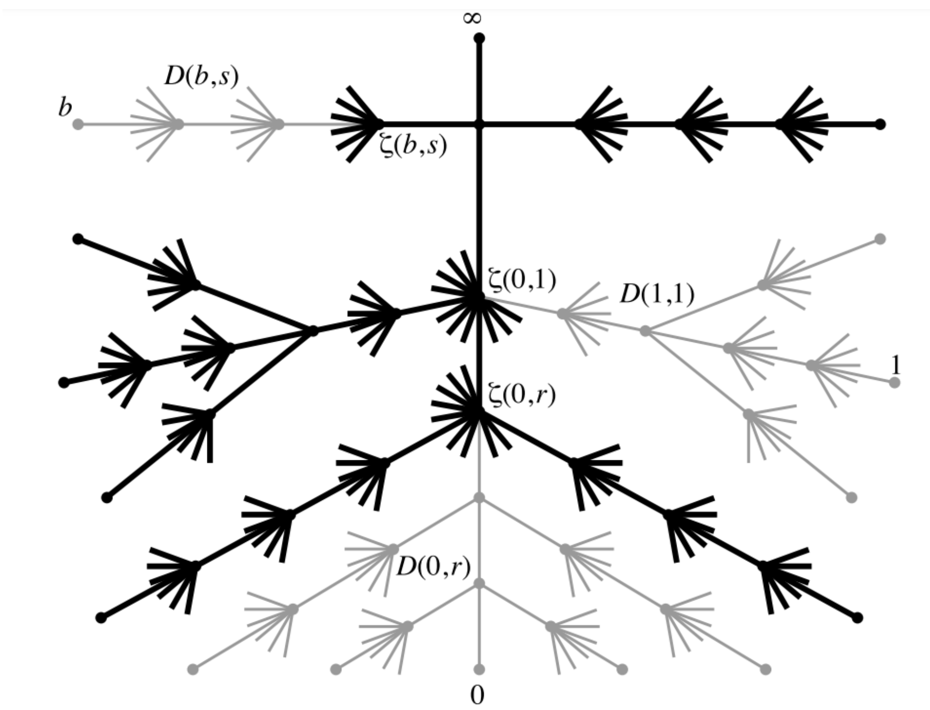
$$D_{an}(a,r) = \{\zeta \in \mathbb{A}^1_{\mathrm{Berk},v} \mid |z-a|_\zeta < r\}$$

be the corresponding *open Berkovich disc.* If $r \notin |\mathbb{C}_v^\times|$, i.e. $\zeta(a,r)$ is a type III point, then $\overline{D}_{an}(a,r) = D_{an}(a,r)$. Otherwise, $\overline{D}_{an}(a,r) \setminus D_{an}(a,r)$ has many points; for example, it contains the type I points z on the circle $|z-a|_v = r$.

5. Show that $\overline{D}_{an}(a,r)$ is closed and $D_{an}(a,r)$ is open in $\mathbb{A}^1_{\mathrm{Berk},v}$.

6. Show that
$$\overline{D}_{an}(a,r) \setminus \{\zeta(a,r)\} = \coprod_{c\in\mathcal{C}} D_{an}(c,r),$$
where $\mathcal{C}$ is a set consisting of one representative for each open disc $D(c,r)$ contained in $D(a,r)$.

The following figure taken from [Ben19] illustrates $\mathbb{A}^1_{\mathrm{Berk},v}$, with three open rational open discs removed. (In this figure, D should be read as D_{an}.)



A basis for the Gelfand topology on $\mathbb{A}^1_{\mathrm{Berk},v}$ can be described as follows: an *open Berkovich disc* is either $D_{an}(a,r)$ for some $r>0$, or the complement of $\overline{D}_{an}(a,r)$ for some $r>0$. An *open connected Berkovich affinoid* is the intersection of finitely many open Berkovich discs.

Theorem 4.1. *The set of open connected Berkovich affinoids in $\mathbb{A}^1_{\mathrm{Berk},v}$ form a basis for the Gelfand topology.*

We now move on to dynamics on $\mathbb{A}^1_{\mathrm{Berk},v}$.

7. Given $f \in \mathbb{C}_v[z]$ and a disc $D(a,r)$, show that $f(D(a,r))$ is also a disc. Hence show that any polynomial $f : \mathbb{C}_v \to \mathbb{C}_v$ extends to a map $f : \mathbb{A}^1_{\mathrm{Berk},v} \to \mathbb{A}^1_{\mathrm{Berk},v}$.

One can define the Fatou and Julia sets for non-archimedean dynamics in analogue to the complex case. Let

$$d_{\mathrm{sph}}([x:y],[u:v]) = \frac{|xv - yu|}{\max\{|x|,|y|\}\max\{|u|,|v|\}}$$

be the spherical metric on $\mathbb{P}^1(\mathbb{C}_v)$. The *classical Fatou set* $\mathcal{F}_{f,I}$ is the set of points $x \in \mathbb{P}^1(\mathbb{C}_v)$ having a neighborhood in which the iterates $\{f^n\}$ are equicontinuous with respect to d_{sph}. The *classical Julia set* is the complement of $\mathcal{F}_{f,I}$.

8. Show that the classical Julia set for $f(z) = z^2$ is empty.

9. Let $\mathbb{C}_v = \mathbb{C}_p$ be the metric completion of the algebraic closure of $\mathbb{Q}_p$ for some rational prime p. Show that the classical Julia set for $f(z) = (z^p - z)/p$ is $\mathbb{Z}_p$.

Now on $\mathbb{A}^1_{\mathrm{Berk},v}$, we say an open set $U \subseteq \mathbb{A}^1_{\mathrm{Berk},v}$ is *dynamically stable* for $f \in \mathbb{C}_v[z]$ if $\bigcup_{n\geq 0} f^n(U)$ omits infinitely many points of $\mathbb{A}^1_{\mathrm{Berk},v}$. The *Berkovich Fatou set* $\mathcal{F}_f$ consists of all points $x \in \mathbb{A}^1_{\mathrm{Berk},v}$ which are dynamically stable for f and the *Berkovich Julia set* is the complement. It is a theorem that the classical Fatou set are exactly the classical (i.e., type I) points that lie in the Berkovich Fatou set.

10. Show that the Berkovich Julia set for $f(z) = z^2$ consists of exactly one point $\zeta(0,1)$.

Problem 12. Potential Theory on $\mathbb{A}^1_{\mathrm{Berk},v}$

To develop a potential theory on $\mathbb{A}^1_{\mathrm{Berk},v}$, we want to define an analog for the Laplacian $\triangle$ on $\mathbb{A}^1_{\mathrm{Berk},v}$. It turns out that we can take advantage of the tree structure of $\mathbb{A}^1_{\mathrm{Berk},v}$, so we will begin our discussion by constructing the Laplacian on trees.

A *finite $\mathbb{R}$-tree* or *finite metrized tree* is a metric space which is homeomorphic to a finite tree (i.e., connected graph with no cycles), such that each edge is identified isometrically with a closed interval in $\mathbb{R}$.

(A word of caution: finite $\mathbb{R}$-trees don't "remember" the underlying graph, and we can freely add or remove degree 2 vertices from the interior of any edge. Hence a quantifier such as "for every edge" should be read as "...under some choice of vertex set/underlying graph".)

Let Γ be a finite $\mathbb{R}$-tree. At every $x \in \Gamma$, there is a set $T_x\Gamma$ of *unit tangent vectors* leading away from x, in one-to-one correspondence with connected components of $\Gamma \setminus \{x\}$. The *directional derivative* of a function $f : \Gamma \to \mathbb{R}$ is

$$d_{\vec{v}} f(x) = \lim_{t\to 0^+} \frac{f(x + t\vec{v}) - f(x)}{t},$$

where this is defined. (The expression $x + t\vec{v}$ above means, for sufficiently small $t > 0$, the point at distance t away from x in the direction $\vec{v}$.) See [BR10, Appendix B] for a more formal discussion and further properties of $\mathbb{R}$-trees.

We will consider the class of *continuous piecewise affine* functions on Γ,

$$\mathrm{CPA}(\Gamma) = \{f : \Gamma \to \mathbb{R} \mid f \text{ continuous and affine linear on each edge}\}.$$

For $f \in \mathrm{CPA}(\Gamma)$, we define

$$\triangle f = \sum_{x \in \Gamma} \left(\sum_{\vec{v} \in T_x \Gamma} d_{\vec{v}} f(x) \right) \delta_x,$$

where δ_x is the delta mass supported at x.

1. Show that if x is in the interior of an edge, then x is not in the support of $\triangle f$.
2. Suppose Γ is a graph with 3 vertices and 2 edges, identified with $[-1, 1]$ with the vertices at $-1, 0$ and 1. Consider $f(x) = 2x$ on $[-1, 0]$ and $f(x) = 3x$ on $[0, 1]$. Compute $\triangle f$.
3. Prove that $\triangle f \equiv 0$ if and only if f is constant.
4. Prove that
$$\int_\Gamma f \triangle g = \int_\Gamma g \triangle f$$
for $f, g \in \mathrm{CPA}(\Gamma)$.

One can easily extend the class of functions to include functions which are C^2 on each open edge and $f'' \in L^1(\Gamma, dx)$ by

$$\triangle f = f''(x)\, dx + \sum_{x \in \Gamma} \left(\sum_{\vec{v} \in T_x \Gamma} d_{\vec{v}} f(x) \right) \delta_x.$$

Now we can equip $\mathbb{A}^1_{\mathrm{Berk},v}$ minus the type I points with the *hyperbolic metric* $d_{\mathbb{H}}$, such that

$$d_{\mathbb{H}}(\zeta(x, r), \zeta(x, R)) = \log R - \log r$$

for any $x \in \mathbb{C}_v$ and $0 < r < R$.

5. Complete the definition of $d_{\mathbb{H}}$.
6. Show that $\triangle \log^+ |z| = \delta_{\zeta(0,1)}$.

In general, one can develop an analog of potential theory on $\mathbb{A}^1_{\mathrm{Berk},v}$ along with a theory of equilibrium measures and Green's functions. In particular,

$$G_f(z) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |f^n(z)|_v$$

extends to a function on $\mathbb{A}^1_{\mathrm{Berk},v}$, whose Laplacian is a measure supported on the Berkovich Julia set J_f, and satisfies analogous properties to the equilibrium measure.

5 Equidistribution for Galois Orbits

We will prove the adelic equidistribution theorem in this section. There are a number of different proofs of it, and we follow the approach by Baker and Rumely [BR06; BR10].

Let K be a number field. For each place $v \in M_K$, let $\mathbb{C}_v$ be the completion of the algebraic closure of K_v, and let $\mathbb{P}^1_{\mathrm{Berk},v}$ be the Berkovich projective line for $\mathbb{C}_v$. A *compact Berkovich adelic set* is a set of the form

$$\mathbb{E} = \prod_{v \in M_K} E_v$$

where E_v is a compact subset of $\mathbb{P}^1_{\mathrm{Berk},v} \setminus \{\infty\}$ and $E_v = \overline{D}(0,1)$ is the closed unit disc for all but finitely many $v \in M_K$. The *logarithmic capacity* of $\mathbb{E}$, denoted by $c(E)$, is defined to be $\prod_{v \in M_K} c(E_v)^{N_v}$ where $N_v = [K_v : \mathbb{Q}_p]/[K : \mathbb{Q}]$.

Let $S \subseteq \overline{K}$ be a $\mathrm{Gal}(\overline{K}/K)$-invariant finite set of points. We define the height of S relative to $\mathbb{E}$ to be

$$h_{\mathbb{E}}(S) = \sum_{v \in M_K} N_v \left(\frac{1}{|S|} \sum_{z \in S} G_v(z) \right)$$

where G_v is the Green's function associated to E_v with respect to ∞.

1. Show that $h_{\mathbb{E}}(S)$ is well-defined, i.e. it is a finite real number.

2. Show that $h_{\mathbb{E}}(S)$ is independent of the choice of embedding $\overline{K} \hookrightarrow \mathbb{C}_v$.

For an element $z \in \overline{K}$, we let S_z be the $\mathrm{Gal}(\overline{K}/K)$-orbit of z and we define $h_{\mathbb{E}}(z) = h_{\mathbb{E}}(S_z)$. The aim of this section is to prove the following adelic equidistribution theorem due to Baker and Rumely:

Theorem 5.1 (Baker–Rumely)**.** *Let K be a number field and $\mathbb{E}$ a compact Berkovich adelic set such that $c(E) = 1$. Suppose $\{z_n\}$ is a sequence of distinct points of $\overline{K}$ such that $h_{\mathbb{E}}(z_n) \to 0$. Fix a place $v \in M_K$ and let δ_n be the uniform probability measure supported on the* $\mathrm{Gal}(\overline{K}/K)$*-orbit of z_n. Then $\{\delta_n\}$ converges weakly to the equilibrium measure μ_v of E_v.*

3. By passing to a subsequence, show we can assume that $\{\delta_n\}$ converges weakly to some probability measure ν.

4. Let U be the connected component containing ∞ of $\mathbb{P}^1_{\mathrm{Berk},v} \setminus E_v$. Show that ν is supported on the complement of U. By replacing E_v with $\mathbb{P}^1_{\mathrm{Berk},v} \setminus U$, show we may assume that ν is supported on E_v.

Let S_n be the $\mathrm{Gal}(\overline{K}/K)$-orbit of z_n. For $v \in M_K$, define

$$D_v(S_n) = \sum_{x,y \in S_n, x \neq y} \log |x - y|$$

and

$$b_{v,n} = N_v(D_v(S_n) + 2G_v(S_n) - c(E_v)).$$

5. Show that $h_{\mathbb{E}}(S_n) \to 0$ implies that $\lim_{n\to\infty} G_v(S_n) = 0$.

6. Show that
$$\sum_{v \in M_K} b_{v,n} = 2h_{\mathbb{E}}(S_n) \geq 0.$$

7. Show that if $\mathbb{E}_v$ is the closed unit disc, then $b_{v,n} \leq 0$.

8. Show that the above three statements imply that $\limsup_{n\to\infty} b_{v,n} = 0$ and so $\limsup_{n\to\infty} D_v(S_n) \geq c(E_v)$.

9. Show that $I(\nu) \geq c(E_v)$ and conclude that ν must be the equilibrum measure of E_v as desired.

We now apply this to the case of polynomials. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a polynomial of degree $d \geq 2$ defined over K. For $v \in M_K$, let E_v be the filled Julia set of f in $\mathbb{P}^1_{\mathrm{Berk},v}$ and let $\mathbb{E} = \prod_{v \in M_K} E_v$.

10. Show that $\mathbb{E}$ is a compact Berkovich adelic set.

We now assume the following theorem:

Theorem 5.2. *Let K be a compact set and $f(z) = a_d z^d + \cdots + a_0$ be a polynomial. Then*
$$c(f^{-1}(K)) = \left(\frac{c(K)}{|a_d|}\right)^{1/d}.$$

11. Show that $c(E_v) = |a_0|_v^{-1/(d-1)}$ and thus $c(\mathbb{E}) = 1$.

12. Show that
$$h_{\mathbb{E}}(f(x)) = dh_{\mathbb{E}}(x)$$
and this is the unique function $h_f : \overline{K} \to \mathbb{R}$ satisfying
$$h_f(f(x)) = dh_f(x) \text{ and } |h_f(x) - h(x)| = O(1)$$
where $h(x)$ is the logarithmic Weil height. We call this the canonical height of f.

13. Show that $h_f(x) = 0$ if and only if x is a preperiodic point of x.

14. Hence show that for each place $v \in M_K$, Galois orbits of preperiodic points converge weakly to the equilibrium measure $\mu_{f,v}$.

6 Applications to Unlikely Intersections

We now prove weaker versions of Theorems 1.1 and 1.2 for polynomials over $\overline{\mathbb{Q}}$ following [BD11]. We start off with Theorem 1.1.

Theorem 6.1. *Fix $a, b \in \overline{\mathbb{Q}}$. The set of parameters $c \in \mathbb{C}$ such that both a and b are preperiodic for $f_c(z) = z^2 + c$ is infinite if and only if $a^2 = b^2$.*

1. For $a \in \mathbb{C}$, define the generalized Mandelbrot set
$$\mathcal{M}_a = \{c \in \mathbb{C} \mid \sup_n |f_c^n(a)| < \infty\}.$$
Let $G_c(z)$ be the Green's/escape-rate function for $f_c(z)$. Show that $G_a(c) = G_c(a^2 + c)$ defines the Green's function for $\mathcal{M}_a$.

2. Show that $\Phi_a(c) = \phi_c(a^2 + c)$ defines a conformal isomorphism in the neighborhood at ∞ where ϕ_c is the Böttcher coordinate for $f_c(z)$.

3. Show that the logarithmic capacity of $\mathcal{M}_a$ is 1.

4. Show that $\mathcal{M}_a = \mathcal{M}_b$ if and only if $a^2 = b^2$. (Hint: Use the fact that ϕ_c is injective.)

Now let a, b be defined in some number field K and let $v \in M_K$ be a non-archimedean place. Then one can let $g_n(T) = f_T^n(a)$ which is a polynomial in T, and then define analogously
$$\mathcal{M}_a = \{c \in \mathbb{A}^1_{\mathrm{Berk},v} \mid \sup_n |g_n(c)| < \infty\} \subseteq \mathbb{A}^1_{\mathrm{Berk},v}.$$

Theorem 6.2. *The logarithimic capacity $c(\mathcal{M}_a)$ is 1 and*
$$G_a(c) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |g_{n+1}(c)|_v$$
defines the Green's function for $\mathcal{M}_a$.

5. Applying the above theorem and the adelic equidistribution theorem, show that if there are infinitely many $c \in \mathbb{C}$ such that a and b are preperiodic for $f_c(z)$, then $\mathcal{M}_a = \mathcal{M}_b$. Hence conclude that $a^2 = b^2$.

We next show the following special case of Theorem 1.2.

Theorem 6.3. *Let $f, g \in \overline{\mathbb{Q}}[z]$ be two polynomials of degree at least two. Then either $|\operatorname{Prep}(f) \cap \operatorname{Prep}(g)|$ is finite or $\operatorname{Prep}(f) = \operatorname{Prep}(g)$.*

Let K be a number field that both f and g are defined over and assume that $|\operatorname{Prep}(f) \cap \operatorname{Prep}(g)|$ is infinite.

6. Show using adelic equidistribution that $\mu_{f,v} = \mu_{g,v}$ for all places $v \in M_K$ where $\mu_{f,v}, \mu_{g,v}$ are the equilibrium measures for f and g at the place v respectively.

7. Hence show that $h_f(z) = h_g(z)$ for all $z \in \overline{\mathbb{Q}}$, where h_f, h_g are the canonical heights associated to f and g respectively.

8. Hence conclude that z is preperiodic for f if and only if z is preperiodic for g.

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