$E/\mathbb{C}$ – elliptic curve

sm. projective. con. alg. gp. dim 1

e.g. $\{y^2z = x^3 + axz^2 + bz^3\} \subset \mathbb{P}^2_{\mathbb{C}}$

$E(\mathbb{C}) \simeq \mathbb{C}/L$

ω – regular 1-form

$L = \langle \int_{\gamma_1} \omega, \int_{\gamma_2} \omega \rangle$, $H_1(E, \mathbb{Z}) = \langle \gamma_1, \gamma_2 \rangle$

$E(\mathbb{C}) \cong (S^1)^2$

$E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$

$E_{tor} \cong (\mathbb{Q}/\mathbb{Z})^2$

$H < E^n$ irr. alg. subgroup

$\psi : H \twoheadrightarrow E^m \quad |\ker\psi| < \infty$

$\therefore \exists\, \phi : E^m \longrightarrow E^n, |\ker\phi| < \infty$

$\operatorname{im}\phi = H$

THM (Raynaud, Manin–Mumford)

Let $V \subset E^n$. Then V contains finitely many maximal torsion cosets.

$E/\overline{\mathbb{Q}}$ $K \subset \overline{\mathbb{Q}}$, E/K.

$$\begin{tikzcd}
\rho_E : G_K \arrow[r] & \mathrm{Aut}(E_{tor}) \arrow[r, "\sim"] \arrow[d, equal] & GL_2(\hat{\mathbb{Z}}) \arrow[d, equal] \\
& \varprojlim \mathrm{Aut}(E[n]) \arrow[r, "\sim"] & \varprojlim GL_2(\mathbb{Z}/n\mathbb{Z})
\end{tikzcd}$$

1. $\mathrm{End}(E) = \mathbb{Z}$

(Serre) $\mathrm{im}\,\rho_E \subset GL_2(\hat{\mathbb{Z}})$
is open.

2. $\mathrm{End}(E) = R$, $\dim_{\mathbb{Z}} R = 2$

$R \subset L = \mathbb{Q}(\sqrt{-d})$

$\mathrm{im}\,\rho_E \subset (R \otimes \hat{\mathbb{Z}})^\times$
is open.

Either way, $\exists H < \hat{\mathbb{Z}}^\times$
open, $H \subset \mathrm{im}\,\rho_E$

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Pfs

intersection

EQUIDISTRIBUTION

$$\left\{\begin{matrix}\text{alg. subgr.}\\ H < E^n\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\text{closed. subs}\\ \text{of } E(\mathbb{C})^n\end{matrix}\right\}$$

G

$\cong (S^1)^{2n}$

Using H, can define

μ_S for real torsion

cosets $S \subset (S^1)^{2n}$

$\{\mu_S, S \subset (S^1)^{2n}\}$ weak-* closed

Pf of MM:

consider $\overline{\left(\bigcup_{\substack{S \subset V \\ \text{torsion} \\ \text{cosets}}} S\right)} = \bigcup_{i=1}^{m} T_i$

T_i - real torsion cosets.

$T_i^{zar} \subset V$

— torsion cosets

Reduction $\mathbb{C} \Longrightarrow \overline{\mathbb{Q}}$

e.g. $E/_{\mathbb{C}} = \{y^2 = x^3 + x + \pi\}$

$$\mathcal{E}/\overline{\mathbb{Q}}(t) = \{y^2 = x^3 + x + t\}$$

non-singular if $4 + 27t^2 \neq 0$.

$$\mathcal{E} \longrightarrow \mathbb{A}^1 - \left\{\pm\sqrt{\frac{-4}{27}}\right\}$$

$$\mathcal{E}_\pi = E.$$

Given $E/\mathbb{C}$, $V \subset E^n$

$\exists$ smooth $\mathcal{B}/\overline{\mathbb{Q}}$,

$p \in \mathcal{B}(\mathbb{C})$

$\mathcal{E}/\mathcal{B}$, $\mathcal{V} \subset \mathcal{E}^n$

$(\mathcal{E}_p, \mathcal{V}_p) = (E, V)$

PF OF MM $\mathbb{C}$

Let $S_1, \ldots, S_n, \ldots \subset V$ be torsion cosets.

$\exists\ S'_1, \ldots, S'_n, \ldots \subset \mathcal{V}$ over B $(S'_j)_p = S_j$.

$p \in \Delta \subset B(\mathbb{C})$

$\Sigma^n_\Delta(\mathbb{C}) \cong (S^1)^{2n} \times \Delta$.

· Let $T_1, \ldots, T_m$ be

s.t. $\bigcup T_i \supset \bigcup S_j$

· T_i - real torsion cosets

· $\bigcup_{i=1}^{m} T_i$ <u>minimum</u>.

BS $MM_{\overline{\mathbb{Q}}}$, $\forall q \in \Delta(\overline{\mathbb{Q}})$

$\bigcup_{i=1}^{m} T_i \subset Y_q$.

· $\bigcup_{i=1}^{m} T_i \subset Y_p = V$

$$\bigcup_{i=1}^{m} T_i^{zar} \subset V$$

André-Oort

$K = \mathbb{Q}(\sqrt{-d}) \supset \mathcal{O}_K$

$$Cl(K) = \frac{\text{non-zero ideals of } \mathcal{O}_K}{I \sim \lambda I,\ \lambda \in K^\times}$$

$$Cl(K) \hookrightarrow Y(1)_{\mathbb{C}} \ni \ldots$$

$\{E : End(E) = \mathcal{O}_K\}$

$$[I] \longrightarrow \mathbb{C}/I$$

$$G_{\mathbb{Q}} \twoheadrightarrow Cl(K) \hookrightarrow \mathrm{AUT}(Y(1)_{CM,\ldots}$$

$$|Cl(K)| = d^{1/2+o(1)}$$

Large Galois orbits.

Special curves are

(i) $T_N \subset Y(1)^2 = Z(\Phi_N)$

(ii) $Y(1)\times\{p\}$, $\{p\}\times Y(1)$, p - CM.

Thm. (André)

$C \subset Y(1)^2$, $|C \cap Y(1)^2_{CM}| = \infty$
irr.

$\Rightarrow C$ is special

$(x, y) \in C$, $x, y \in Y(1)_{CM, d}$

$\exists\ \mathfrak{p} \subset \mathcal{O}_K$, prime,
$[\mathcal{O}_K : \mathfrak{p}] = p$

$\therefore ([\mathfrak{p}] \cdot x, [\mathfrak{p}] \cdot y) \in G_{\mathbb{Q}} \cdot (x, y)$

?

$C \cap (T_p \times T_p) C$

<u>Need</u>: $p \le d^{1/4 + o(1)}$

$\Longleftarrow$ GRH