Classification theory, stability and analyticity

B.Zilber

Arizona Winter School

March 4, 2023

Structures and languages

Structures are given as

$$\mathrm{M} := (M; \sigma), \quad \sigma \text{ a vocabulary (signature, language)}$$

e.g.

$$\mathbb{C}_{\text{field}} := (\mathbb{C}; +, \cdot), \text{ the field of complex numbers.}$$

Note, that the metric is not definable in $\mathbb{C}_{\text{field}}$.

Classification Theory

Aim: classify (theories of) structures with respect to their definability (expressibility) properties in the given language.

Classification Theory

Aim: classify (theories of) structures with respect to their definability (expressibility) properties in the given language.

In particular, the property of $\mathrm{Th}(M)$ to define its model of cardinality κ uniquely up to isomorphism: κ**-categoricity**.

Classification Theory

Aim: classify (theories of) structures with respect to their definability (expressibility) properties in the given language.

In particular, the property of $\mathrm{Th}(\mathrm{M})$ to define its model of cardinality κ uniquely up to isomorphism: κ**-categoricity**.

In Fact (Morley, 1965) $\kappa_1 > \aleph_0$ and $\kappa_2 > \aleph_0$ then κ_1-categoricity is equivalent to κ_2-categoricity.

Classification Theory

An example: $\mathrm{ACF}_0 := \mathrm{Th}(\mathbb{C}_{\text{field}})$ is uncountably categorical.

Classification Theory

An example: $\mathrm{ACF}_0 := \mathrm{Th}(\mathbb{C}_{\mathrm{field}})$ is uncountably categorical.

Corollary. Given a complex algebraic variety V over k and σ_{Zar}= the collection of Zariski closed subsets (m-ary relations) on V^m defined over k, the structure

$$\mathrm{V}_{\mathrm{Zar}} = (V; \sigma_{\mathrm{Zar}})$$

is categorical.

Classification Theory

Categorical structures (theories) are at the very **top** of the classification hierarchy.

Classification Theory

Categorical structures (theories) are at the very **top** of the classification hierarchy.

Further on we have (the hierarchy of tameness):
- ω-stable of finite rank
- ω-stable
- superstable
- stable
- $\ldots$

Classification Theory

Categorical structures (theories) are at the very **top** of the classification hierarchy.

Further on we have (the hierarchy of tameness):
- ω-stable of finite rank
- ω-stable
- superstable
- stable
- ...

o-minimal form a side-branch of the classification theory.

Stability

A catchphrase to characterise the model-theoretic Stability Theory: *A generalisation of Algebraic Geometry*:

Stability

A catchphrase to characterise the model-theoretic Stability Theory: *A generalisation of Algebraic Geometry*:

- We have an analogue of the *dimension of a variety*, the (Morley) rank of a definable set.

Stability

A catchphrase to characterise the model-theoretic Stability Theory: *A generalisation of Algebraic Geometry*:

- We have an analogue of the *dimension of a variety*, the (Morley) rank of a definable set.
- Analogue of the *transcendence degree*.

Stability

A catchphrase to characterise the model-theoretic Stability Theory: *A generalisation of Algebraic Geometry*:

- We have an analogue of the *dimension of a variety*, the (Morley) rank of a definable set.
- Analogue of the *transcendence degree*.
- Analogue of an irreducible curve, a *strongly minimal set*: every definable subset of the set is either finite or the complement to a finite.

Stability

A catchphrase to characterise the model-theoretic Stability Theory: *A generalisation of Algebraic Geometry*:

- We have an analogue of the *dimension of a variety*, the (Morley) rank of a definable set.
- Analogue of the *transcendence degree*.
- Analogue of an irreducible curve, a *strongly minimal set*: every definable subset of the set is either finite or the complement to a finite.

An important example beyond AG is the structure (ω-stable of rank ω) *differentially closed field* DCF_0:

$$(\mathrm{F}; +, \cdot, D), \quad D \text{ a differential operator.}$$

Beyond algebraic geometry. Analyticity.

Can model theory develop a formal analogue of analytic geometry, as an extension of algebraic geometry?

Beyond algebraic geometry. Analyticity.

Can model theory develop a formal analogue of analytic geometry, as an extension of algebraic geometry?

An example:

$$\mathbb{C}_{\mathrm{exp}} := (\mathbb{C}; +, \cdot, \mathrm{exp}).$$

Is it *quasi-minimal*: every definable subset is either countable or the complement to a countable?

Beyond algebraic geometry. Analyticity.

Can model theory develop a formal analogue of analytic geometry, as an extension of algebraic geometry?

An example:

$$\mathbb{C}_{\mathrm{exp}} := (\mathbb{C}; +, \cdot, \mathrm{exp}).$$

Is it *quasi-minimal*: every definable subset is either countable or the complement to a countable?
The question is open since 1991.

Beyond algebraic geometry. Analyticity.

Can model theory develop a formal analogue of analytic geometry, as an extension of algebraic geometry?

An example:

$$\mathbb{C}_{\text{exp}} := (\mathbb{C}; +, \cdot, \text{exp}).$$

Is it *quasi-minimal*: every definable subset is either countable or the complement to a countable?
The question is open since 1991.

Ultimately, the “tameness” of $\mathbb{C}_{\text{exp}}$ is formulated as a categoricity statement of an L_{ω,ω_1}-axiom system.

Pseudo-exponentiation

Theorem (2003-2011). There is an axiom system Σ_{exp} (not first-order) such that Σ_{exp} has a unique model

$$\mathrm{K}_{\mathrm{exp}}(\kappa) = (\mathrm{K}; , +, \cdot, \mathrm{exp})$$

in every uncountable cardianlity κ.
$\mathrm{K}_{\mathrm{exp}}(\kappa)$ is quasi-minimal.
Σ_{exp} contains the Schanuel conjecture and an Nullstellensatz (exp-Nullstellensatz).

Pseudo-exponentiation

Theorem (2003-2011). There is an axiom system Σ_{exp} (not first-order) such that Σ_{exp} has a unique model

$$\mathrm{K}_{\text{exp}}(\kappa) = (\mathrm{K};\, , +, \cdot, \text{exp})$$

in every uncountable cardianlity κ.
$\mathrm{K}_{\text{exp}}(\kappa)$ is quasi-minimal.
Σ_{exp} contains the Schanuel conjecture and an Nullstellensatz (exp-Nullstellensatz).

Conjecture. $\mathbb{C}_{\text{exp}} \cong \mathrm{K}_{\text{exp}}(\kappa)$, for $\kappa =$ continuum.

A test for the "tame exp"-conjecture

Theorem. (2002: Wilkie, Koiran, Z.) There is an entire complex function f satisfying a "Schanuel conjecture for f" for any finite $X \subset \mathbb{C}$,

$$\mathrm{SC}_f : \ \mathrm{tr.deg}(X \cup f(X)) - \mathrm{size}(X) \geq 0.$$

and an f-Nullstellensatz.

A test for the “tame exp”-conjecture

Theorem. (2002: Wilkie, Koiran, Z.) There is an entire complex function f satisfying a “Schanuel conjecture for f” for any finite $X \subset \mathbb{C}$,

$$\mathrm{SC}_f: \ \mathrm{tr.deg}(X \cup f(X)) - \mathrm{size}(X) \geq 0.$$

and an f-Nullstellensatz.
The structure

$$\mathbb{C}_f = (\mathbb{C}; +, \cdot, f)$$

is quasiminimal and can be categorically axiomatised by some axioms Σ_f.

A test for the "tame exp"-conjecture (continued)

$\mathbb{C}_f$ satisfies the following f**-Nullstellensatz**: Let $W \subseteq \mathbb{C}^{2n}$ be an irreducible algebraic variety in variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ s.t.

$$\exists x_i, y_i \bigwedge_{i<j\leq n} x_i \neq x_j \ \& \ \langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle \in W$$

and, for any $1 \leq i_1 < \ldots i_k \leq n$,

$$\dim \mathrm{pr}_{i_1 \ldots i_k, i_1 \ldots i_k} W \geq k$$

(projection onto $\langle x_{i_1}, \ldots, x_{i_k}, y_{i_1}, \ldots, y_{i_k} \rangle$-space).
Then there is a point

$$\langle a_1, \ldots, a_n, f(a_1), \ldots, f(a_n) \rangle \in W.$$

A test for the “tame exp”-conjecture (continued)

The axiom(s) SC_f are first-order axiomatisable and this implies that $\mathrm{Th}(\mathbb{C}_f)$ is ω-stable.

A test for the "tame exp"-conjecture (continued)

The axiom(s) SC_f are first-order axiomatisable and this implies that $\mathrm{Th}(\mathbb{C}_f)$ is ω-stable.

Remark. The statement "SC_f is first-order axiomatisable" is equivalent to the trivial ZP-conjecture:

A test for the "tame exp"-conjecture (continued)

The axiom(s) SC_f are first-order axiomatisable and this implies that $\mathrm{Th}(\mathbb{C}_f)$ is ω-stable.

Remark. The statement "SC_f is first-order axiomatisable" is equivalent to the trivial ZP-conjecture: "special" subsets of $\mathbb{C}^n$ are given by equations $x_i = x_j$.

A test for the “tame exp”-conjecture (continued)

The axiom(s) SC_f are first-order axiomatisable and this implies that $\mathrm{Th}(\mathbb{C}_f)$ is ω-stable.

Remark. The statement “SC_f is first-order axiomatisable” is equivalent to the trivial ZP-conjecture: “special” subsets of $\mathbb{C}^n$ are given by equations $x_i = x_j$.

Exercise. The statement “$\mathrm{SC}_{\exp}$ is first-order axiomatisable” is equivalent to the ZP-conjecture for $\mathbb{G}_m$.

Raising to irrational powers in $\mathbb{C}$

Let $r_1, \ldots, r_m \in \mathbb{C}$ and read

$$\mathbb{C}^{r_1,\ldots,r_m} = (\mathbb{C}; +, \cdot, X^{r_1}, \ldots, X^{r_n})$$

where X^r stands for the multivalued operation (relation)

$$y = \exp(r \ln x).$$

Raising to irrational powers in $\mathbb{C}$

Let $r_1, \ldots, r_m \in \mathbb{C}$ and read

$$\mathbb{C}^{r_1,\ldots,r_m} = (\mathbb{C}; +, \cdot, X^{r_1}, \ldots, X^{r_n})$$

where X^r stands for the multivalued operation (relation)

$$y = \exp(r \ln x).$$

Proposition (2015) *The statement "$SC_{\mathbb{C}^{r_1,\ldots,r_m}}$ is first-order axiomatisable" is equivalent to the Mordell-Lang statement for $\mathbb{G}_m$ (M.Laurent's theorem).*

Raising to irrational powers in $\mathbb{C}$

Let $r_1, \ldots, r_m \in \mathbb{C}$ and read

$$\mathbb{C}^{r_1,\ldots,r_m} = (\mathbb{C}; +, \cdot, X^{r_1}, \ldots, X^{r_n})$$

where X^r stands for the multivalued operation (relation)

$$y = \exp(r \ln x).$$

Proposition (2015) *The statement "$SC_{\mathbb{C}^{r_1,\ldots,r_m}}$ is first-order axiomatisable" is equivalent to the Mordell-Lang statement for $\mathbb{G}_m$ (M.Laurent's theorem).*

Theorem (F.Gallinaro, 2022) $\mathbb{C}^{r_1,\ldots,r_m}$*-Nullstellensatz is valid unconditionally.*
The proof is based on tropical geometry techniques.

Constructions and fine classification theory

Three classical dimension notions:

$$\text{tr.deg}, \;\; \text{lin.dim}, \;\;\; \text{size}.$$

Constructions and fine classification theory

Three classical dimension notions:

$$\text{tr.deg}, \ \text{lin.dim}, \quad \text{size}.$$

A present day reading of The Trichotomy Conjecture (1983) is:

The three classical dimensions are the only ones that can occur in stable structures.

Constructions and fine classification theory

Three classical dimension notions:

$$\text{tr.deg}, \ \text{lin.dim}, \ \ \text{size}.$$

A present day reading of The Trichotomy Conjecture (1983) is:

The three classical dimensions are the only ones that can occur in stable structures.

Hrushovski's construction (1989): one can mix the three dimension notions to construct new ones fitting the criteria of stability (and even categoricity).

Hrushovski predimension and Fraisse amalgamation

Example (1990). Suppose we have two field structures on the same set F :

$$(F; +_1, \cdot_1) \text{ and } (F; +_2, \cdot_2).$$

Hrushovski predimension and Fraisse amalgamation

Example (1990). Suppose we have two field structures on the same set F :

$$(F; +_1, \cdot_1) \text{ and } (F; +_2, \cdot_2).$$

We can then consider a **predimension** notion: for each finite X

$$\delta(X) := \mathrm{tr.deg}_1(X) + \mathrm{tr.deg}_2(X) - \mathrm{size}(X)$$

Hrushovski predimension and Fraisse amalgamation

Example (1990). Suppose we have two field structures on the same set F :

$$(F; +_1, \cdot_1) \text{ and } (F; +_2, \cdot_2).$$

We can then consider a **predimension** notion: for each finite X

$$\delta(X) := \mathrm{tr.deg}_1(X) + \mathrm{tr.deg}_2(X) - \mathrm{size}(X)$$

Let $\mathcal{F}$ be the class of all such $(F; +_1, \cdot_1, +_2, \cdot_2)$ which satisfy the *Hrushovski predimension inequality*

$$\delta(X) \geq 0 \text{ for any finite } X \subset F.$$

Hrushovski predimension and Fraisse amalgamation

Example (1990). Suppose we have two field structures on the same set F :

$$(F; +_1, \cdot_1) \text{ and } (F; +_2, \cdot_2).$$

We can then consider a **predimension** notion: for each finite X

$$\delta(X) := \mathrm{tr.deg}_1(X) + \mathrm{tr.deg}_2(X) - \mathrm{size}(X)$$

Let $\mathcal{F}$ be the class of all such $(F; +_1, \cdot_1, +_2, \cdot_2)$ which satisfy the *Hrushovski predimension inequality*

$$\delta(X) \geq 0 \text{ for any finite } X \subset F.$$

Theorem (Hrushovski) One can amalgamate structures in $\mathcal{F}$. There is $\mathrm{F} \in \mathcal{F}$ which is strongly minimal (and so categorical) and has a dimension notion δ^* different from the classical ones.

Hrushovski construction a step further

More generally, the fusion of two classical structures

$$(M_1; \mathcal{L}_1) \text{ and } (M_2; \mathcal{L}_2)$$

by the fusing map $f : M_1 \twoheadrightarrow M_2$ and a predimension

$$\delta_f(X) = d_1(X) + d_2(f(X)) - d_3(X) \geq 0$$

where d_1, d_2, d_3 classical.

Hrushovski construction a step further

More generally, the fusion of two classical structures

$$(M_1; \mathcal{L}_1) \text{ and } (M_2; \mathcal{L}_2)$$

by the fusing map $f : M_1 \twoheadrightarrow M_2$ and a predimension

$$\delta_f(X) = d_1(X) + d_2(f(X)) - d_3(X) \geq 0$$

where d_1, d_2, d_3 classical.
E.g. $M_1 = M_2 = \mathbb{C}_{\mathrm{field}}$, fused by $\exp : \mathbb{C} \to \mathbb{C}$,

$$\delta_{\exp}(X) = \mathrm{tr.deg}(X \cup \exp X) - \mathrm{lin.dim}(X) \geq 0$$

Hrushovski construction a step further

All known examples of tame analytic structures have been explained by Hrushovski predimension theory.