3.1 Abelian varieties & curves with cyclic action

(not so) hidden agenda:

Do smooth curves w/ interesting NP's exist?

Does open Torelli locus intersect NP strata that have small dim?

smooth curves

Jacobians

3.1 arithmetic — yes!

geometry — well....

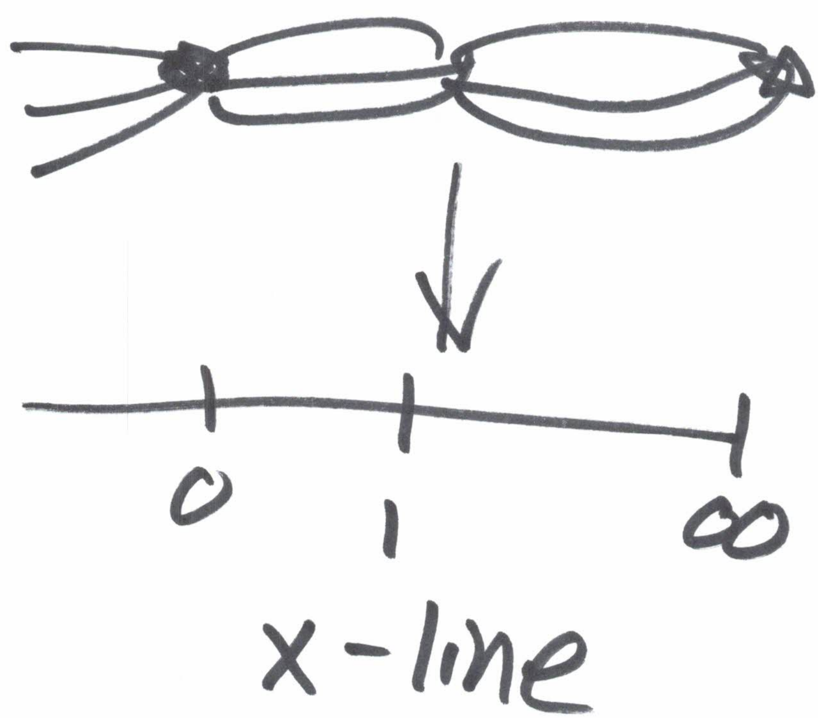
today: $g=6$

History:

$C: y^m = x(x-1)$

m odd prime

quotient of Fermat curve



3.1 $\quad g_C = \left(\frac{m-1}{2}\right)$

$$\Upsilon(x,y) = (x, \zeta_m y)$$

order m

$$\mathbb{Q}(\zeta_m) = K \circlearrowright \mathrm{Jac}(C)$$

/ deg $m-1$

$\mathbb{Q}$ $\quad$ has complex multiplication.

3.1 Weil

Jacobi sums

f order of p mod m

f even $\Longrightarrow C$ supersingular

<u>Ex.</u> $m=13$ $g=6$

ss $\cancel{\Rightarrow}$ if $p \not\equiv 1, 3, 9$ mod 13

3.2 μ_m - action

Cyclic μ_m - cover

$$C \xrightarrow{} \mathbb{P}^1$$
μ_m

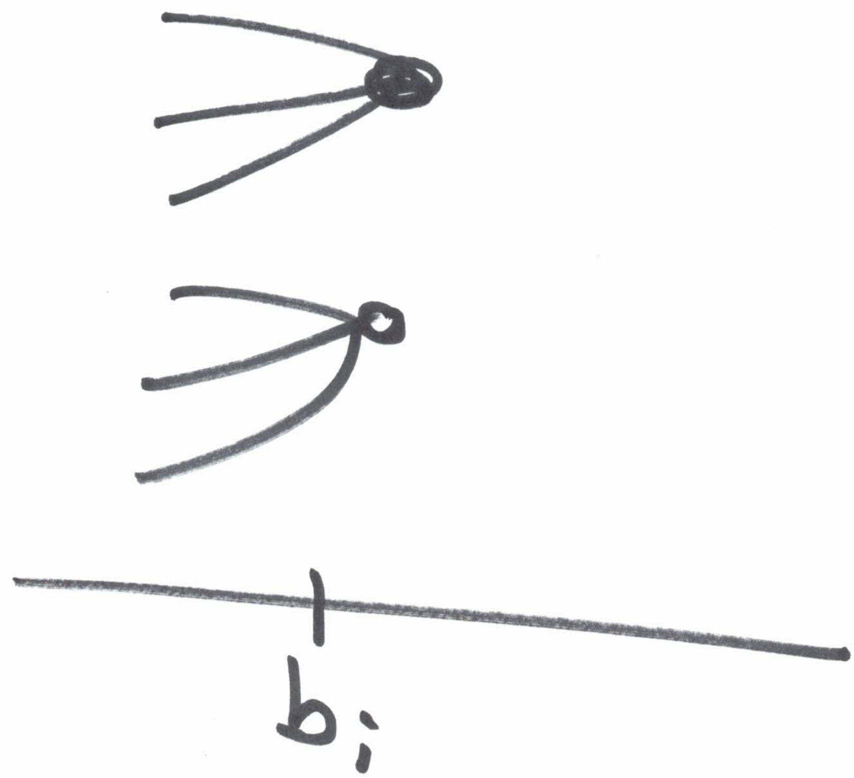
m = degree

$$y^m = \prod_{i=1}^{N} (x - b_i)^{a_i}$$

branch point

$$\sum a_i \equiv 0 \mod N$$

N = # branch



move 3 branch points
to $0, 1, \infty$

3.2 Ex.

$M[16]$: $m=5$, $N=5$, $g=6$

$$y^5 = x(x-1)(x-b_1)(x-b_2)$$

$M[17]$ $m=9$, $N=4$, $g=7$

$$y^9 = x(x-1)(x-b_1)$$

3.3

$$1 \le a_i < m$$

$$\sum a_i \equiv 0 \mod m$$

$$\vec{a} = (a_1, \dots a_N)$$

inertia type

$$\gamma = (m, N, \vec{a})$$

monodromy data

3.3 $\mathcal{H}_\gamma$ Hurwitz space

$$C \xrightarrow[\mu_m]{} \mathbb{P}^1$$

$\#$branch $= N$

inertia $\vec{a}$

$$\begin{array}{c} \mathcal{H}_\gamma \\ \downarrow \\ \mathcal{M}_g \end{array}$$

$\dim(\mathcal{H}_\gamma) =$

$N - 3$

3.4 Abelian varieties

μ_m - action

$$H^0(C, \Omega^1) = \bigoplus L_i$$

$\uparrow$ acts on L_i by mult by ζ_m^i

$\vec{a}$ determines

$f_i = \dim(L_i)$

$\vec{f} = (f_1, \ldots f_{m-1})$

signature type

Ex M[16] $m=5$

$\vec{f} = (3,2,1,0)$

3.4

$$S_\gamma = \left\{ \begin{array}{l} X \text{ ab. var} \\ \mu_m\text{-action} \\ \text{signature } \vec{f} \end{array} \right\}$$

moduli space

3.4

$$\begin{tikzcd}
\mathcal{H}_\gamma \arrow[r] \arrow[d] & S_\gamma \arrow[d] \\
\mathcal{M}_g \arrow[r, "T"] & \mathcal{A}_g
\end{tikzcd}$$

$C \to \mathbb{P}^1$

μ_m

$\mathrm{Jac}(C)$

$\circlearrowleft$

μ_m

S_γ Deligne-Mostow Shimura variety.

3.5 $m = \text{odd}$

$$\dim(S_g) = \sum_{i=1}^{\frac{m-1}{2}} f_i f_{m-i}$$

$$
\begin{tikzcd}
\mathcal{H}_\gamma \arrow[r, "T_\gamma"] \arrow[d] & \mathcal{S}_\gamma \arrow[d] \\
\mathcal{M}_g \arrow[r] & \mathcal{A}_g
\end{tikzcd}
$$

dim linear in g

dim quad-ing

easier to study

$g = 2, 3$
image open & dense
false $g \geq 4$
is $\mathrm{Im}(T_\gamma)$ open & dense in S_γ?

"Def" γ is <u>Special</u> if image of T_γ open & dense in S_γ

"almost every" abelian var $X \circlearrowleft \mu_m$ sig. f is a Jacobian.

3.5 / $\mathbb{C}$

Oort's expectation:

$\underline{g \geq 8}$ ~~or not specia~~

Coleman conj. CM abelian varieties

false for 5-7

3.6 Mod p

Restrictions on p-rank

NP & EO type

for $C \in \mathcal{H}_g$

or $X \in \mathcal{S}_g$

Understood on

$\mathcal{S}_g$

3.6. $[\times p]$ on $\mathbb{Z}/m - \{0\}$

orbits of action

$m = 5 \quad p \equiv 2 \mod 5$

$$\begin{tikzcd}
3 \arrow[r] & 1 \arrow[r] & 2 \arrow[d] \\
 & & 4 \arrow[ull, bend left]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{Z}[\zeta_m] \arrow[d, dash] \\
\langle p \rangle
\end{tikzcd}$$

Ex. M_{16} $\quad (3, 2, 1, \underline{\underline{0}})$

dimensions

$[x_p] \curvearrowright \{L_1, L_2, L_3, L_4\}$

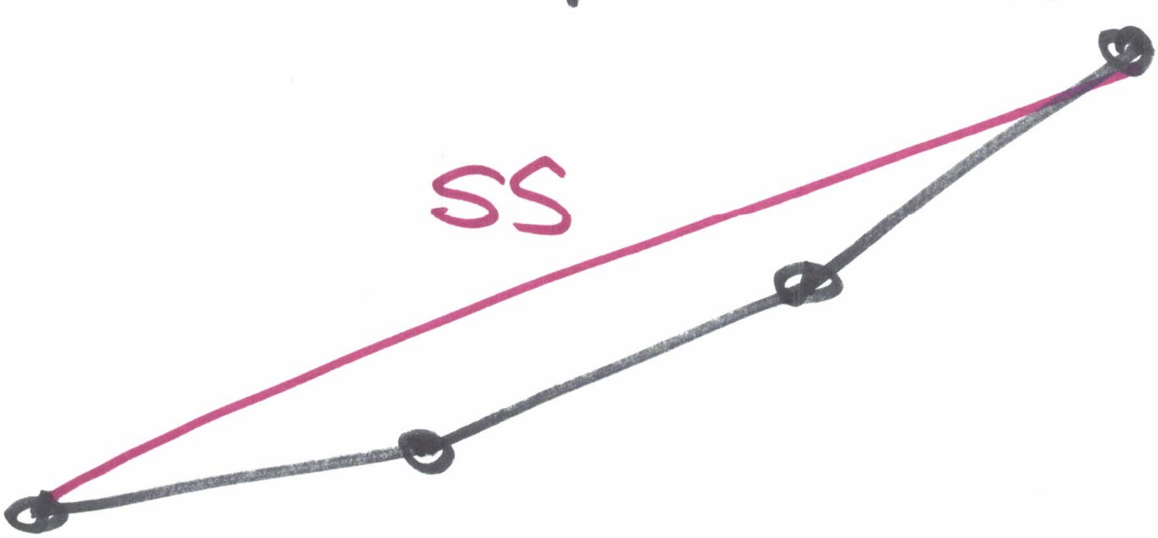
$p \equiv 2 \bmod 5$

max p-rank is 0

conditions on NP

$\Longrightarrow G_{1,3} \oplus G_{3,1} \oplus G_{1,1}^2$

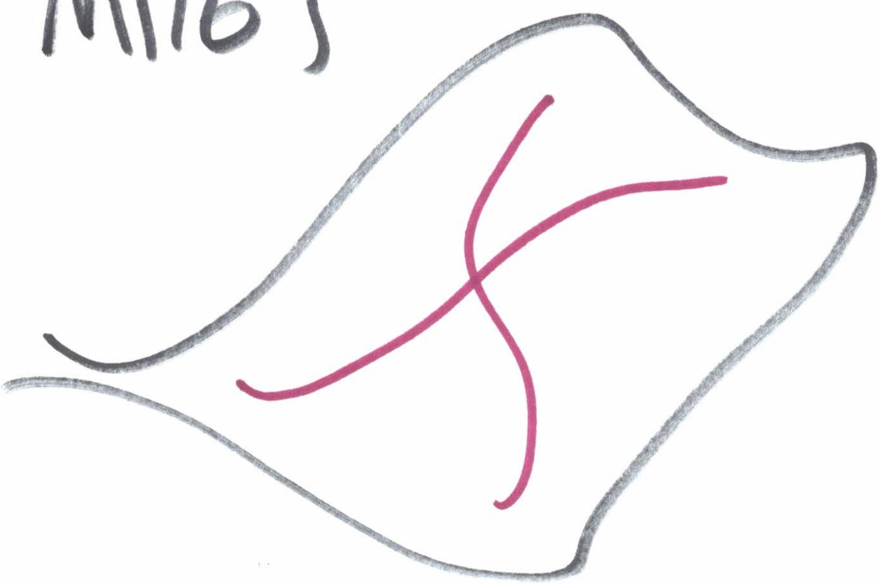
slopes $(1/4, 3/4)$ + $(1/2, 1/2)$



3.8 Li / Mantovan / Tang / P

M[16]

M[19]



App $p >> 0$

M[16] $\exists$ smooth supersing. curve genus $g = 6$ $\forall$ $p \equiv 2, 3, 4 \bmod 5$

M[19] $\exists$ . . . . $g = 7$ $p \equiv 2 \bmod 3$

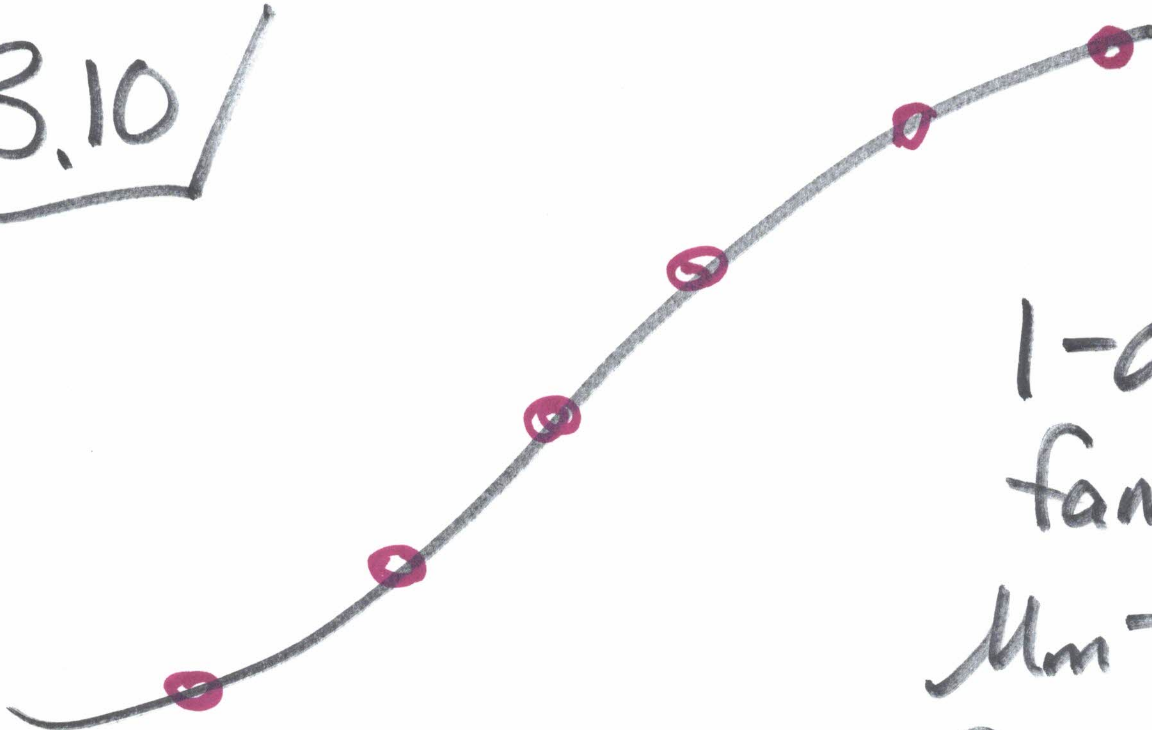
3.9 | not special

Miller/Katz/Kotwitz:

Image of $\mathcal{M}_g$ in $\mathcal{A}_g$
intersects { ord locus
non-ord locus

LMPT: infinitely many
types of γ
image of $\mathcal{H}_\gamma$ in S_γ
intersects { μ-ord locus
non-μ-ord locus

3.10



1-dim family of μ_m-covers $C \to \mathbb{P}^1$

$N=4$

Q: rate of growth of # of non μ-ord curves in family as p grows.

w/ Renzo Cavalieri