4.1 Families of abelian varieties & curves : ℓ-adic monodromy

<u>Theme</u>

geometric techiques $\rightsquigarrow$ arithmetic applications

1) dim of families
   decomp of Jacobians
2) Complete families & boundary
3) extra automorphisms
   - Hurwitz spaces
   - Shimura varieties
4) monodromy

existence of curves w/ unusual Newton polygons.

4.2 ℓ primes $k = \bar{k}$ $\mathrm{char}(k) = p$

$\ell \neq p$

X p.p. abvar dim g

$X[\ell](k) \cong (\mathbb{Z}/\ell)^{2g}$ ← pick basis

ℓ-torsion

ℓ^n $(\mathbb{Z}/\ell^n)^{2g}$

ℓ-adic Tate module

$$\left\{ \begin{array}{c} D \\ \text{unram } \mu_\ell\text{-covers} \downarrow \\ C \\ \text{genus } g \end{array} \right\} \longleftrightarrow \text{bijection}$$

$$\longrightarrow J_C$$

4.2 cont

$\alpha \in \pi_1(W, w$

4.3

$\pi_1(W,w)$		$M \in GL_{2g}$
α	$\longrightarrow$	change of basis on $(\mathbb{Z}/\ell)^{2g}$ matrix

monodromy representation

$$\pi_1(W,w) \longrightarrow \cancel{G}Sp_{2g}(\mathbb{Z}/\ell)$$

W has big monodromy!

if image is $Sp_{2g}(\mathbb{Z}/\ell)$

$\boxed{4.4}$

Strategy for showing big monodromy

find

→ elements that generate $Sp_{2g}(\mathbb{Z}/\ell)$

Hall –

not possible today

→ subgroups

Van Kampen

4.4 cont.

Thm Chai $W \subset \mathcal{A}_g$ irred

s.t. [W stable under Hecke correspondences

& generic point of W not supersingular $W \not\subset \mathcal{A}_g[ss]$

then W has big monodromy!

true for all NP strata of $\mathcal{A}_g$ other than ss.

Torelli locus not stable under Hecke corres. unless $g = 2, 3$

4.5 $\quad F = \mathbb{F}_{p^a} \quad k = \overline{F}$

<u>Thm</u> Chai $\quad W$ family of curves ~~or abelian varieties~~ with big monodromy.

(i) $\exists\ C/_k$ in W s.t. $\mathrm{Aut}(C) = \{1\}$

(ii) $\exists\ C/_k$ in W s.t. $\mathrm{Jac}(C) =$ absolutely simple

(iii) if $|\mathbb{F}| \equiv 1 \bmod \ell$ then about ℓ/ℓ^2-1 of the $\mathbb{F}$-curves in W have a point of order ℓ in $\mathrm{Jac}(C)(\mathbb{F})$

(iv) for most $\mathbb{F}$-curves in W

$L(C/\mathbb{F}, T) \leftarrow$ poly of deg $2g$ in $\mathbb{Z}[T]$

$K \leftarrow$ splitting field

$\mathbb{Q} \qquad \deg(K/\mathbb{Q}) = 2^g g!$

4.6 / Joint w/ J. Achter

Fix p. Fix $0 \le f \le g$

$W = \mathcal{M}_g^f$ p-rank $\le f$ stratum of $\mathcal{M}_g$

$C \in W$ $\# J_C[p](k) \le p^f$

Motivation

Faber & Van der Geer

Thm $\dim(\mathcal{M}_g^f) = (3g-3) - (g-f)$
$= 2g - 3 + f$

$\mathcal{M}_g^g$

Applications: typical curve C of

	p-rank 0	supersingular
(i)	$\mathrm{Aut}(C)=\{1\}$	expected true $g=3$ Conj: Oort
(ii)	$\mathrm{Jac}(C)$ abs. simple	false $\mathrm{Jac}(C) \sim E^g$
(iii)	~~$\sim 1/\ell$ pr curve~~ prob of having ℓ-torsion pt $\rightsquigarrow 1/\ell$	?
(iv)	splitting field L-poly big	false:

$g \geq 3$

Biggest difference

mod ℓ - monodromy

for supersingular locus

trivial!

4.7 $g \geq 4$

Q: Is M_g^f irreducible?

<u>Thm</u> Achter-P

$g \geq 3$

$0 \leq f \leq g$

W irred comp of M_g^f

then big monodromy!

mod-ℓ $Sp_{2g}(\mathbb{Z}/\ell)$

ℓ-adic $Sp_{2g}(\mathbb{Z}_\ell)$

$\boxed{\text{or if } g=2 \text{ then } f \neq 0}$

<u>Thm</u>: Version for $\mathcal{H}_g^f$

<u>4.8</u>/ proof big-monodromy

~~for~~ $Sp_{2g}(\mathbb{Z}/\ell)$

for $\mathcal{M}_g^f$

$g=3$

$$\mathcal{M}_3^f \underset{\substack{\text{open}\\ \text{dense}}}{\subset} \mathcal{A}_3^f$$

apply Chai Thm

issue: $g=2$ $f=0$

$$\mathcal{M}_2^0 \subset \mathcal{A}_2[ss]$$

not big monodromy

strategy: induction

avoid $g=2$

$g=1$

4.9 proof cont.

~~FVdG~~ :

FVdG $\Rightarrow \dim(M_g^f) = 2g-3+f$

$W \subset M_g^f$ will intersect

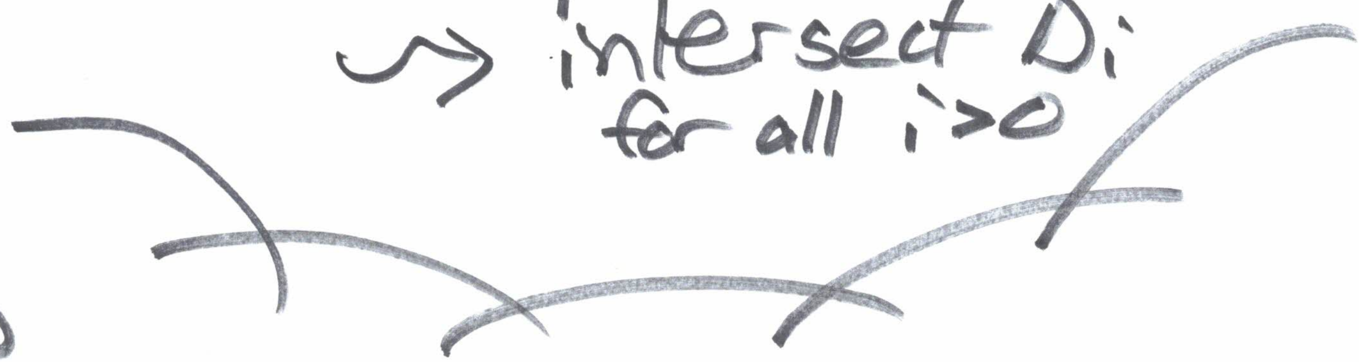
Δ_i : boundary comp of $\overline{M}_g$

for some i

$f=0$

$\overline{W}$ contains points like.

$\rightsquigarrow$ intersect Δ_i for all $i>0$



A-P generalized:

f : $\overline{W}$ contains chains of elliptic curves

choose locations of ord elliptic curves in chain

4.9 __proof__ claim mod ℓ-monodromy N_{ℓ} is $Sp_{2g}(\mathbb{Z}/\ell)$

true for $g \leq 3$

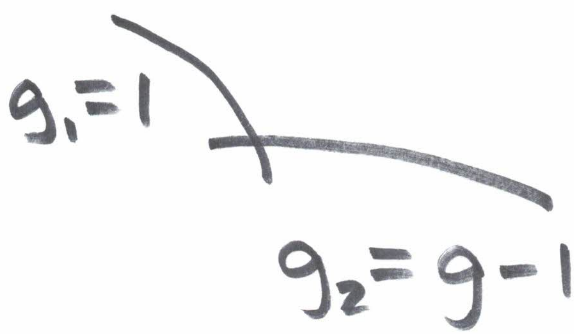
Avoid $g=4$

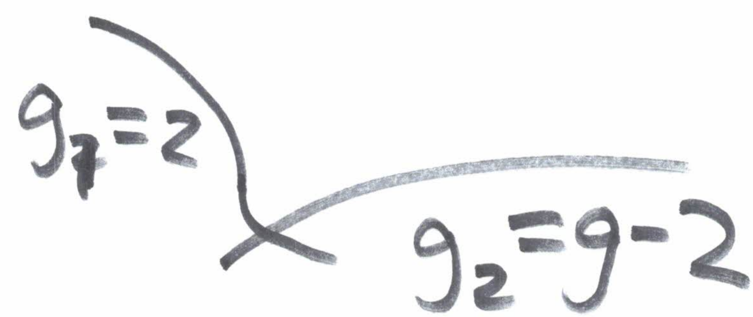
$g \geq 5$ $\quad W$ component of $\mathcal{M}_g^f$

$\overline{W}$ intersects D_i

$i=1$ & $i=2$

inductive hyp.





$$\pi : \mathcal{M}_{g_1,1}^{f_1} \times \mathcal{M}_{g_2,1}^{f_2} \to \overline{W}$$

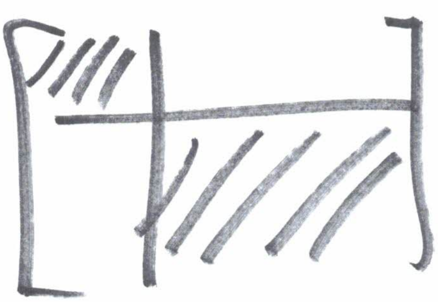
<u>step</u> $\overline{W}$ contains full image of clutching map for ~~same~~ any f_1, f_2 w/ $f_1 + f_2 = f$

+ ~~at~~ any g_1, g_2 w/ $g_1 + g_2 = g$.

N_ℓ contains $\cancel{Sp_2(\mathbb{Z}/\ell)} \times Sp_{2g-2}(\mathbb{Z}/\ell)$

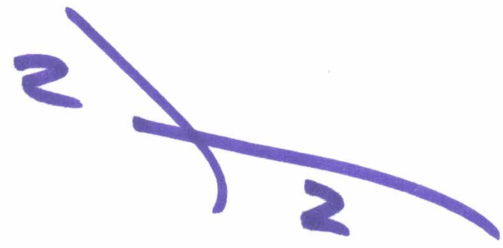
$\cancel{Sp_4(\mathbb{Z}/\ell)} \times Sp_{2g-4}(\mathbb{Z}/\ell)$

this part not accurate when $f_1 = 0$ but it doesn't matter.



conclude group theory!

what about $g=4$?



$M_2^0 \subset A_2[SS]$

intersect $\Delta_{1,1}$

two different copies of

$Sp_6(\mathbb{Z}/\ell)$ $\cancel{Sp_4(\mathbb{Z}/\ell) \times Sp_4(\mathbb{Z}/\ell)}$

in N_ℓ.

Mistake! I meant to write 2 copies of $Sp_6(\mathbb{Z}/\ell)$ that are different

$\mathcal{M}_g^0$ p-rank 0
dim $2g-3$
non-empty

$g \geq 5$
$\mathcal{M}_g[ss]$ not known
if non-empty.

<u>Oort's Expectation</u>

as $\dim(\mathcal{M}_g) <$
$\operatorname{codim}(\mathcal{A}_g[ss], \mathcal{A}_g)$

$$3g-3 < g\frac{(g+1)}{2} - \lfloor g^2/4 \rfloor$$

$g \geq 9$ unlikely intersection of $\mathcal{M}_g$ w/ $\mathcal{A}_g[ss]$

<u>Oort's Conj</u>

supersingular curves exist $\forall g, \forall p$