3. Lecture 3: Lower Bounds for Canonical Heights

Today we'll be moving from K to $\bar{K}$, so we need to normalize our heights appropriately. This can be done (see notes) and gives height functions

$$h : \bar{K} \longrightarrow [0, \infty),$$
$$\hat{h}_D : A(\bar{K}) \longrightarrow [0, \infty).$$

[We always take $D \in \mathrm{Div}(A)$ ample and symmetric.]

$\hat{h}_D$ is "canonical" for $A(\bar{K})$. Similarly h is "canonical" for the group $\mathbb{G}_m(\bar{K}) = \bar{K}^*$.

$$\hat{h}_D : A(K) \to [0, \infty) \quad \text{satisfies} \quad \hat{h}_D\bigl([m]P\bigr) = m^2 \hat{h}_D(P).$$
$$h : \mathbb{G}_m(K) \to [0, \infty) \quad \text{satisfies} \quad h(\alpha^m) = |m| \cdot h(\alpha).$$

Where do they vanish?

$$\hat{h}_D(P) = 0 \iff P \in A(\bar{K})_{\mathrm{tors}}$$
versus
$$h(\alpha) = 0 \iff \alpha \in \mathbb{G}_m(\bar{K})_{\mathrm{tors}} = \{\text{roots of unity}\}.$$

This raises a fundamental question:

How small can the canonical height be, if it's not zero?

3.1. Small Heights for a Fixed A and Varying Fields.

Intuition: Small non-zero canonical height requires a large field extension.

Example 1: For $\mathbb{G}_m(\bar{\mathbb{Q}})$,

$$h(2^{1/n}) = \frac{1}{n} h(2), \quad \text{but} \quad \bigl[\mathbb{Q}(2^{1/n}) : \mathbb{Q}\bigr] = \#\boldsymbol{\mu}_n = n.$$

Example 2: Similarly, for $P \in A(K)$, we can take $Q \in [n]^{-1}(P)$ to get

$$\hat{h}_D(Q) = \frac{1}{n^2} \hat{h}_D(P), \quad \text{but} \quad \bigl[K(Q) : K\bigr] \approx \#A[n] = n^{2g}.$$

(Classical) Lehmer Conjecture: There is a constant $C > 0$ such that

$$h(\alpha) \ge \frac{C}{\bigl[\mathbb{Q}(\alpha) : \mathbb{Q}\bigr]} \quad \text{for all } \alpha \in \mathbb{G}_m(\bar{\mathbb{Q}}) \smallsetminus \mathbb{G}_m(\bar{\mathbb{Q}})_{\text{tors}}.$$

Best proven result:
Theorem (Dobrowolski, 1979):

$$h(\alpha) \ge \frac{C}{d} \cdot \left(\frac{\log \log d}{\log d} \right)^3 \quad \begin{array}{l} \text{for all } \alpha \in \mathbb{G}_m(\bar{\mathbb{Q}}) \smallsetminus \mathbb{G}_m(\bar{\mathbb{Q}})_{\text{tors}} \\ \text{where } d = \bigl[\mathbb{Q}(\alpha) : \mathbb{Q}\bigr]. \end{array}$$

Lehmer Conjecture for Abelian Varieties: (Masser 1984): Let $g = \dim(A)$. There is a constant $C(A/K, D) > 0$ such that

$$\hat{h}_D(P) \ge \frac{C(A/K)}{\bigl[K(P) : K\bigr]^{1/g}} \quad \text{for all } P \in A(\bar{K}) \smallsetminus A(\bar{K})_{\text{tors}}.$$

Some partial results (due to a bunch of different people, see notes):

Theorem: Let

$$d = d(K, P) = \bigl[K(P) : K\bigr], \qquad C = C(A/K, D, \epsilon).$$

Then

(a) $\hat{h}_D(P) \ge C/d^{2g+1+\epsilon}$ for all A/K.
(b) $\hat{h}_D(P) \ge C/d^{1+\epsilon}$ if A/K has CM.
(c) $\hat{h}_D(P) \ge C/d^{2+\epsilon}$ if $g = 1$ and $j(A) \notin R_K$.

There are two methods that have been used for Lehmer's problem (called informally):

1. **Transcendence Theory Method**
2. **Fourier Averaging Method**

Transcendence Theory Method: Let L/K with $d = [L : K]$. Look at

$$A(L, B) := \bigl\{P \in A(L) : \hat{h}_{A,D}(P) \le B\bigr\}.$$

Goal is to show that

$$\#A\left(L, \frac{C_1}{d}\right) \le C_2 d^{g+\epsilon}.$$

Exploit group law by considering

$$A(L,B)^{(g)} := \bigl\{P_1 + \cdots + P_g : P_1,\ldots,P_g \in A(L,B)\bigr\},$$

so

$$\#A(L,B)^{(g)} \approx \frac{\#A(L,B)^g}{g!} \quad \text{lots of points,}$$

$$A(L,B)^{(g)} \subseteq A(L,g^2B) \quad \text{with height not too large.}$$

Then

(1) Construct a non-zero "small" (theta) function F on A that vanishes to high order at the points in $A(L,B)^{(g)}$.

(2) Use Cauchy's theorem to get upper bound for partial derivatives $\bigl|\partial F(Q)\bigr|$ for $Q \in A(L,B)^{(g)}$.

(3) Use a zero-estimate from transcendence theory to get a lower bound for partial derivatives $\bigl|\partial F(Q)\bigr|$ for $Q \in A(L,B)^{(g)}$.

(4) If $\#A(L,B)$ is sufficiently large, the upper and lower bounds contradict.

Note: Many details have been omitted!!

Fourier Averaging Method:

Up to now, primarily applied to $\dim(A) = 1$. [If time at end, say a few words about this.]

3.2. Small Heights for a Fixed K and Varying A/K.

Returning to our fundamental question:

How small can the canonical height be, if it's not zero?

We move in an orthogonal direction:

Intuition: Small non-zero canonical height of a point requires a "complicated" abelian variety.

First step: How to measure the "complexity" (height) of an abelian variety. Some possibilities:

- Define $h(A/K)$ to be the smallest height of the coefficients of polynomials that describe a projective embedding of A/K. E.g. For an elliptic curve

$$E : y^2 = x^3 + Ax + B, \quad \text{set}$$
$$h(E/K) := \min_{u \in K^*} h\bigl([1, u^4 A, u^6 B]\bigr).$$

- Let $\mathcal{A}_g \subset \mathbb{P}^N$ be the moduli space of principally polarized abelian varieties of dimension g with a projective embedding. Then define

$$h(A/K) := h_{\mathcal{A}_g}\bigl(j(A)\bigr) + \log \mathsf{N}_{K/\mathbb{Q}}\bigl(\text{Conductor}(A/K)\bigr).$$

  Here $j(A) \in \mathcal{A}_g(K)$ is the moduli point associated to A.

Note: We need the conductor in (2) to deal with twists, i.e., abelian varieties with the same $j(A)$.

Dem'janenko–Lang Height Conjecture:
Let $P \in A(K)$ satisfy $\mathbb{Z} \cdot P$ is Zariski dense in A. Then

$$\hat{h}_D(P) \geq C_1(K,g) \cdot h(A/K) - C_2(K,g).$$

Some Partial Results (by various people):

(I) The DLH conjecture is true if $j(A)$ is at least ϵ-distance away from the boundary of $\mathcal{A}_g(\mathbb{C})^{\text{simple}}$.

(II) The DLH conjecture is true for twists, i.e., for a fixed value of $j(A)$.

(III) For $\dim(A) = 1$, the DLH conjecture is true for A with bounded Szpiro ratio $\frac{\log|\text{Disc}|}{\log|\text{Cond}|}$. In particular,

$$abc\text{-conjecture} \quad \Longrightarrow \quad \text{DLH conjecture for elliptic curves.}$$

A brief word about the proofs:

(I) Transcendence theory method.

(II) K-rational points on twists give points in fields with large discriminant on original abelian variety.

(III) Fourier averaging method.