4. Lecture 4: Canonical Heights in Families & Specialization Theorems

Example: A family of elliptic curves and points:

$$E_T : y^2 = x^3 + T^2 x - 1, \quad P_T = (1, T).$$

We can plug in values $T \in \mathbb{Q}$ and compute (using PARI, where $D = 2(O)$):

t	0	2	17	1729	22/7	355/113
$\hat{h}_{E_t}(P_t)$	0	0.93	2.51	7.11	3.24	5.68

Questions:

- Is P_t a non-torsion point for all $0 \neq t \in \mathbb{Q}$?
- How does $\hat{h}_{E_t}(P_t)$ vary as a function of $t \in \mathbb{Q}$?

For a general result, we need some preliminary setup.

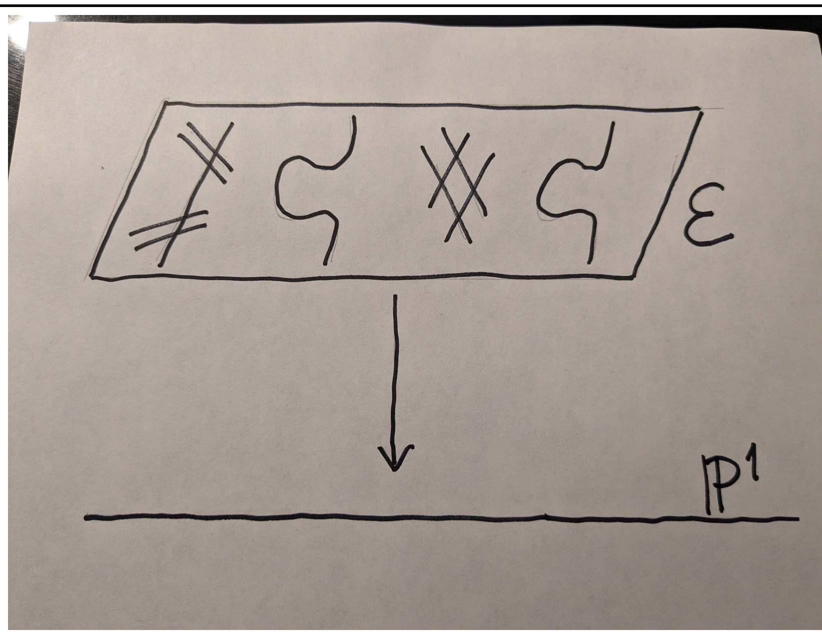

Figure 1. A family of elliptic curves

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K	a number field.
C/K	a smooth projective curve C/K.
$A/K(C)$	an abelian variety defined over $K(C)$.
$(\mathcal{A},\pi)$	a family of abelian varieties $\pi : \mathcal{A} \to C$ with generic fiber is A.
P	a point $P \in A\bigl(K(C)\bigr)$.
$\mathcal{P}$	the associated section $\mathcal{P} : C \to \mathcal{A}$.

Definition: For $t \in C(\bar{K})$, the associated *specialization map* is

$$\mathsf{S}_t : A\bigl(\bar{K}(C)\bigr) \longrightarrow \mathcal{A}_t(\bar{K}), \quad \mathsf{S}_t(P) = \mathcal{P}_t.$$

Specialization Theorem: Assume that $A/\bar{K}(C)$ has no "constant part," i.e., no part coming from an abelian variety $B/\bar{K}$. Then there is a constant H_0 such that

$$t \in C(\bar{K}) \text{ and } h_C(t) \ge H_0 \quad \Longrightarrow \quad \mathsf{S}_t : A\bigl(\bar{K}(C)\bigr) \to \mathcal{A}_t(\bar{K}) \text{ is injective.}$$

The proof uses:

Height Limit Theorem: Let $D \in \operatorname{Div}(A/K)$, and let $\mathcal{D} \in \operatorname{Div}(\mathcal{A}/K)$ be its closure. Fix a Weil height function h_C on $C(\bar{K})$ associated to a divisor of degree 1. Then

$$\lim_{\substack{t \in C(\bar{K}) \\ h_C(t) \to \infty}} \frac{\hat{h}_{\mathcal{A}_t,\mathcal{D}_t}(\mathcal{P}_t)}{h_C(t)} = \hat{h}_{A,D}(P). \qquad (*)$$

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Proof Sketch Height Theorem $\Rightarrow$ Specialization Theorem: We have several heights and height pairings:

- Function field canonical height $\hat{h}_{A,D}$ on $A\bigl(\bar{K}(C)\bigr)$.
- Number field canonical heights $\hat{h}_{A_t,D_t}$ on each fiber $A_t(\bar{k})$.
- Number field height on $C(\bar{K})$.

The theorem gives the formula

$$\lim_{\substack{t\in C(\bar{K})\\ h_C(t)\to\infty}} \frac{\langle P_t, Q_t\rangle_{\mathcal{A}_t,\mathcal{D}_t}}{h_C(t)} = \langle P, Q\rangle_{A,D}.$$

Let $P_1,\dots,P_r \in A\bigl(\bar{K}(C)\bigr)$ generate modulo torsion. Then

$$\lim_{\substack{t\in C(\bar{K})\\ h_C(t)\to\infty}} \frac{\operatorname{Reg}_{\mathcal{D}_t}\bigl(\mathsf{S}_t(P_1),\dots,\mathsf{S}_t(P_r)\bigr)}{h_C(t)^r} = \underbrace{\operatorname{Reg}_D(P_1,\dots,P_r)}_{\text{Positive since } P_1,\dots,P_r \text{ independent.}} > 0.$$

Hence

$$\begin{aligned} &h_C(t) \text{ sufficiently large} \\ &\qquad \Longrightarrow \quad \mathsf{S}_t(P_1),\dots,\mathsf{S}_t(P_r) \text{ are independent.} \end{aligned}$$

(Additional argument to deal with torsion part of $A\bigl(\bar{K}(C)\bigr)$.)

Generalizations and Strengthenings:

- *Higher dimensional bases*: Consider $\mathcal{A} \to B$ with $\dim(B) \geq 2$.

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- *Rank Jumps*: We proved

$$\operatorname{rank} \mathcal{A}_t(K) \ge \operatorname{rank} A\bigl(K(C)\bigr) \quad \text{for } t \in C(K),\ h_C(t) \gg 1.$$

How frequently can the rank of $\mathcal{A}_t(K)$ be strictly larger? By how much?

- *Unlikely Intersections*: If $\dim(A) \ge 2$, a dimension count suggests that there is a <u>finite</u> set $\Sigma \subset C(\bar{K})$ such that

$$t \in C(\bar{K}) \smallsetminus \Sigma \quad \Longrightarrow \quad \mathsf{S}_t : A\bigl(\bar{K}(C)\bigr) \to \mathcal{A}_t(\bar{K}) \text{ is injective.}$$

- *Improved Asymptotics*: We proved

$$\hat{h}_{\mathcal{A}_t,\mathcal{D}_t}(\mathcal{P}_t) = \hat{h}_{A,D}(P) \cdot h_C(t) + o\bigl(h_C(t)\bigr).$$

Various people have shown that one can replace the $o\bigl(h_C(t)\bigr)$ with:
– $O\bigl(h_C(t)^{2/3}\bigr)$ in general.
– $O\bigl(h_C(t)^{1/2}\bigr)$ if $C = \mathbb{P}^1$ or $\dim(A) = 1$.
– $O(1)$ if $C = \mathbb{P}^1$ and $\dim(A) = 1$.

Proof Sketch of the Height Limit Theorem: (as time allows)

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We start with the triangle inequality

$$\left|\hat{h}_{\mathcal{A}_t,\mathcal{D}_t}(\mathcal{P}_t) - \hat{h}_{A,D}(P)\cdot h_C(t)\right| \tag{a}$$

$$\leq \left|\hat{h}_{\mathcal{A}_t,\mathcal{D}_t}(\mathcal{P}_t) - h_{\mathcal{A},\mathcal{D}}(\mathcal{P}_t)\right| \tag{b}$$

$$+ \left|h_{\mathcal{A},\mathcal{D}}(\mathcal{P}_t) - h_{A,D}(P)\cdot h_C(t)\right| \tag{c}$$

$$+ \left|h_{A,D}(P) - \hat{h}_{A,D}(P)\right|\cdot h_C(t). \tag{d}$$

For (b), we know in general that

$$\hat{h}_{\mathcal{A}_t,\mathcal{D}_t} = h_{\mathcal{A},\mathcal{D}} + O(1),$$

but the $O(1)$ depends on t. One can make the t-dependence explicit (interesting argument using blow-up to resolve a rational map, see notes):

$$\hat{h}_{\mathcal{A}_t,\mathcal{D}_t} = h_{\mathcal{A},\mathcal{D}} + O\big(h_C(t)\big) \quad \text{on } \mathcal{A}_t(\bar{K}), \tag{b$'$}$$

where the big O constant does not depend on t.

The key to estimating (c) is to use the fact that

$$\mathcal{P} : C \longrightarrow \mathcal{A}$$

is a morphism. (Here is where we use $\dim(C) = 1$.) So by functoriality of heights:

$$h_{\mathcal{A},\mathcal{D}}(\mathcal{P}_t) = h_{C,\mathcal{P}^*\mathcal{D}}(t) + O_{\mathcal{P}}(1) \quad \text{for } t \in C(\bar{K}). \tag{c$'$}$$

For (d), the function field version says

$$\hat{h}_{A,D} = h_{A,D} + O(1) \quad \text{on } A\big(\bar{K}(C)\big), \tag{d$'$}$$

Substituting (b′), (c′), and (d′) into (a) and dividing by $h_C(t)$ yields

$$\left|\frac{\hat{h}_{\mathcal{A}_t,\mathcal{D}_t}(\mathcal{P}_t)}{h_C(t)} - \hat{h}_{A,D}(P)\right| \le \left|C_1 + \frac{C_2}{h_C(t)}\right| + \left|\frac{h_{C,\mathcal{P}^*\mathcal{D}}(t)}{h_C(t)} + \frac{C_3(P)}{h_C(t)} - h_{A,D}(P)\right| + C_4. \quad \text{(e)}$$

For any effective $\Delta_1, \Delta_2 \in \operatorname{Div}(C)$, we have (another height property)

$$\lim_{\substack{t\in C(\bar{K})\\ h_C(t)\to\infty}} \frac{h_{C,\Delta_1}(t)}{h_{C,\Delta_2}(t)} = \frac{\deg(\Delta_1)}{\deg(\Delta_2)},$$

and hence

$$\lim_{\substack{t\in C(\bar{K})\\ h_C(t)\to\infty}} \frac{h_{C,\mathcal{P}^*\mathcal{D}}(t)}{h_C(t)} = \underbrace{\deg(\mathcal{P}^*\mathcal{D}) = h_{A,D}(P) + O(1)}_{\text{Height via intersection theory over } \bar{K}(C).}.$$

Using this in (e) yields

$$\limsup_{\substack{t\in C(\bar{K})\\ h_C(t)\to\infty}} \left|\frac{\hat{h}_{\mathcal{A}_t,\mathcal{D}_t}(\mathcal{P}_t)}{h_C(t)} - \hat{h}_{A,D}(P)\right| \le C_5. \quad \text{(f)}$$

Key Observation: The constant C_5 does not depend on P. We know that

$$\hat{h}_{\mathcal{A}_t,\mathcal{D}_t}\big([m]\mathcal{P}_t\big) = m^2 \cdot \hat{h}_{\mathcal{A}_t,\mathcal{D}_t}(\mathcal{P}_t),$$
$$\hat{h}_{A,D}\big([m]P\big) = m^2 \cdot \hat{h}_{A,D}(P),$$

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so replacing P by $[m]P$ in (f) gives

$$\limsup_{\substack{t\in C(\bar{K})\\ h_C(t)\to\infty}} \left| \frac{\hat{h}_{\mathcal{A}_t,\mathcal{D}_t}(\mathcal{P}_t)}{h_C(t)} - \hat{h}_{A,D}(P) \right| \le \frac{C_5}{m^2} \quad \text{for all } m \ge 1.$$

Let $m \to \infty$ to complete the proof.

Mathematics Department, Box 1917, 151 Thayer Street, Brown University, Providence, RI 02912 USA
Email address: joseph_silverman@brown.edu

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