Math 124 - Exam 3 Practice problems- Spring '06

1. A window frame has the shape of a rectangle with a semicircle on top. All four sides of the rectangle are part of the frame. The straight portions of the frame (the four sides of the rectangle) cost 8$ per foot, and the curved portion (the semicircle) costs 12$ per foot. The total area of the window must be 20 square feet. Minimize the cost of the frame.

Let $x$ be the width of the frame, $y$ the height of the rectangular part. So the radius of the semicircle is $x/2$. The cost is

$$C = 8(2y + 2x) + 12\left(\frac{\pi x}{2}\right) = 16y + 16x + 6\pi x$$

The area must be 20, so

$$20 = xy + \frac{1}{2}\pi \left(\frac{x}{2}\right)^2 = xy + \frac{1}{8}\pi x^2$$

Solve this for $y$:

$$y = \frac{20 - \frac{1}{8}\pi x^2}{x}$$

Then plug this in for $y$ in the formula for $C$:

$$C = 16\frac{20 - \frac{1}{8}\pi x^2}{x} + 16x + 6\pi x$$

$$= \frac{320}{x} - 2\pi x + 16x + 6\pi x = \frac{320}{x} + 16x + 4\pi x$$

(1)

Set $dC/dx = 0$

$$0 = \frac{-320}{x^2} + 16 + 4\pi$$

(2)

So

$$x = \sqrt{\frac{320}{16 + 4\pi}} \approx 3.347$$

(3)
This gives \( y \approx 4.661 \). The cost is \( C = 16y + 16x + 6\pi x = $191.22 \).

2. \( f(x) = xe^{-x^2} \)
   (a) Find all \( x \) values at which \( f \) has a local min or max.
   \( f \) has a local min at \(-1/\sqrt{2}\) and a local max at \(1/\sqrt{2}\).
   (b) Find all inflection points.
   \( f \) has inflection points at \( x = 0, \sqrt{3/2}, -\sqrt{3/2} \).
   (c) Find the global min and max over \( 0 \leq x \leq 1 \).
   The only critical point inside this interval is \( 1/\sqrt{2} \). So the global min and max will occur at \( x = 0, 1/\sqrt{2} \) or \( 1 \). \( f(0) = 0, f(1/\sqrt{2}) = e^{-1/2}/\sqrt{2} \approx 0.4288 \) and \( f(1) = e^{-1} \approx 0.3678 \). So the global min of \( f \) is 0 and is attained at \( x = 0 \) and the global max of \( f \) is \( e^{-1/2}/\sqrt{2} \approx 0.4288 \), and is attained at \( x = 1/\sqrt{2} \).

3. \[
\lim_{x \to \infty} \frac{1 - \cos(ax)}{x^2},
\]
This is NOT a L’Hopital problem. The numerator is bounded between 0 and 2 and the denominator goes to \( \infty \), so the fraction converges to 0.

\[
\lim_{x \to 0} \frac{1 - \cos(ax)}{x^2},
\]
The numerator and denominator both go to 0, so we can apply L’Hopital. In fact, you have to apply it twice:
\[
= \lim_{x \to 0} \frac{a \sin(ax)}{2x} = \lim_{x \to 0} \frac{a^2 \cos(ax)}{2} = \frac{a^2}{2}
\]

\[
\lim_{x \to \infty} \frac{\ln(x)}{\sinh(x)}
\]
The numerator and denominator both go to \( \infty \), so we apply L’Hopital:
\[
= \lim_{x \to \infty} \frac{1/x}{\cosh(x)}
\]
Now the numerator goes to zero and the denominator goes to $\infty$. So the fraction goes to 0.

4. Let $f(x) = 5a^3x^2 - 2x^5$. Here $a$ is a parameter; it does not depend on $x$.
(a) Find the critical points and determine if they are local min or maxs. Your answer should involve $a$.

$$f'(x) = 10a^3x - 10x^4 = 10x(a^3 - x^3)$$

So we have critical points at $x = 0$ and $x = a$.

First consider the case of $a > 0$. For $x < 0$ it is easy to see $f'(x) < 0$. And when $x \to \infty$, $f'(x)$ is negative. For $0 < x < a$, $a^3 - x^3$ will be positive, so $f'(x)$ will be positive. So by the first derivative test, $x = 0$ is a local min, $x = a$ is a local max.

Now consider the case of $a < 0$. So the critical point at $a$ is now left of the one at 0. For $x > 0$, $a^3 - x^3$ will be negative, so $f'(x)$ will be negative. when $x \to \infty$, $f'(x)$ is negative. So $x = a$ will be a local min and $x = 0$ will be a local max.

Finally consider the case that $a = 0$. Then $f(x) = -2x^5$. This has one critical point at $x = 0$ and it is neither a local min or max.

(b) Find the global max over $x \geq 0$.

First note that as $x \to \infty$, $f(x)$ goes to $-\infty$, regardless of the value of $a$. So this will never be the global max. If $a > 0$ we have to compare $f(0) = 0$ and $f(a) = 3a^5$. So the global max is $3a^5$ and it is attained at $x = a$.

If $a < 0$ we need only consider $x = 0$. $f(0) = 0$, so the global max is 0 and it is attained at $x = 0$.

If $a = 0$ the global max is 0 and it is attained at $x = 0$.

To check your answers, graph the function for two values of $a$, one positive and one negative.

5. Consider the curve $x^3 + y^2 \cosh(y - 1) = 2$
(a) Find the equation of the tangent line at the point $(1, 1)$ to the curve.

Implicit differentiation:

$$3x^2 + 2y \cosh(y - 1) \frac{dy}{dx} + y^2 \sinh(y - 1) \frac{dy}{dx} = 0$$

Plug in $x = 1$ and $y = 1$,

$$3 + 2 \cosh(0) \frac{dy}{dx} + \sinh(0) \frac{dy}{dx} = 0$$

$$3 = 0$$
Using \( \cosh(0) = 1, \sinh(0) = 0 \),

\[
3 + 2 \frac{dy}{dx} = 0
\]

So \( \frac{dy}{dx} = -3/2 \). The tangent line must go through \((1, 1)\), so its equation is

\[
y - 1 = -3/2(x - 1), \text{ or } y = -3x/2 + 5/2.
\]

(b) Use your answer to (a) to find approximately the value of \( y \) so that \((1.01, y)\) is also on the curve.

Use the tangent line to approximate the curve: \( y = -1.5 \times 1.01 + 5/2 = 0.985 \).