Math 129 (Fall '05) - Exam 2 - Kennedy

1. (15 points) Consider

$$y = 1 + \sin x, \quad 0 \leq x \leq 2\pi \tag{1}$$

(a) Graph y over the interval $[0, 2\pi]$.
(b) The area below the graph of y, above the x-axis and between $x = 0$ and $x = 2\pi$ is rotated about the x-axis. Find the volume of the resulting solid.
Solution: Slice it perpendicular to the x-axis. The slices are discs, so their volume is $\pi r^2 \Delta x$. Here $r = y = 1 + \sin x$. So the total volume is

$$\int_0^{2\pi} \pi(1 + \sin x)^2 \, dx = \pi \int_0^{2\pi} (1 + 2\sin x + \sin^2 x) \, dx \tag{2}$$

The last integral can be done using the tables to give

$$\pi \left[x - 2\cos x + \frac{1}{2} \sin x \cos x + \frac{x}{2} \right]_0^{2\pi} = 3\pi^2 \tag{3}$$

2. (12 points) Consider the improper integral:

$$\int_0^2 \frac{e^{-x}}{x^{1/8}} \, dx \tag{4}$$

(a) Determine whether the improper integral converges and explain your reasoning. **Solution:** On the interval of integration we have

$$\frac{e^{-x}}{x^{1/8}} \leq \frac{1}{x^{1/8}} \tag{5}$$

You can do the integral for

$$\int_0^2 \frac{1}{x^{1/8}} \, dx \tag{6}$$

to see that this integral converges. So by the comparison theorem the original integral converges.
(b) Use your calculator to estimate the value of the integral to two decimal places. Note that if you just run the program with $A = 0$ and $B = 1$ the calculator will give you an error. **Solution:** You need to run the program

with A slightly greater than 0. Make a table of several values of A and the calculator result for the integral from A to the upper limit.

$$\begin{aligned} A &= 0.01, SIMPSON(50) = 1.0278, SIMPSON(100) = 1.0275 \\ A &= 0.005, SIMPSON(50) = 1.0374, SIMPSON(100) = 1.0367 \\ A &= 0.0001, SIMPSON(50) = 1.0578, SIMPSON(100) = 1.0517 \\ A &= 0.00005, SIMPSON(50) = 1.0608, SIMPSON(100) = 1.0532 \\ A &= 0.00001, SIMPSON(50) = 1.0686, SIMPSON(100) = 1.0571 \end{aligned} \tag{7}$$

So it is about 1.05. For the integral from 0 to 2 it is about 0.97.

3. (12 points) Find the sums of the following series **Solution:** Depending on your exam, you had some subset of the following:

$$1 + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{30}} = \frac{1 - \frac{1}{2^{31}}}{1 - 1/2} = 2 - \frac{1}{2^{30}} \tag{8}$$

$$x^2 - x^3 + x^4 - x^5 + \cdots = x^2(1 - x + x^2 - x^3 + \cdots) = \frac{x^2}{1 + x} \tag{9}$$

$$1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + + \frac{1}{3^8} + \cdots = \frac{1}{1 - 1/3^2} = \frac{9}{8} \tag{10}$$

$$1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + + \frac{1}{2^8} + \cdots = \frac{1}{1 - 1/2^2} = \frac{4}{3} \tag{11}$$

$$1 + \frac{1}{3^1} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{30}} = \frac{1 - \frac{1}{3^{31}}}{1 - 1/3} = \frac{3}{2}(1 - \frac{1}{3^{31}}) \tag{12}$$

4. (12 points) Consider the graph of $y = \sqrt{4 - x^2}$ for $-2 \leq x \leq 2$. (Other exam had $y = \sqrt{9 - x^2}$ for $-3 \leq x \leq 3$.)
(a) Find an integral that gives the length of this graph. (You don't have to evaluate the integral.) **Solution:**

$$y' = \frac{-x}{\sqrt{4 - x^2}} \tag{13}$$

So the length is

$$\int_{-2}^{2} \sqrt{1+(y')^2} = \int_{-2}^{2} \sqrt{1+\frac{x^2}{4-x^2}} = \int_{-2}^{2} \sqrt{\frac{4}{4-x^2}} \tag{14}$$

For the other exam you get

$$\int_{-3}^{3} \sqrt{\frac{9}{9-x^2}} \tag{15}$$

(b) Use your calculator to approximate your integral to two decimal places.
(c) Explain why your answer to (b) is close to 2π. **Solution:** The graph is the graph of a top half of a circle of radius 2. So the full circumference would be $2\pi r = 4\pi$ and so half of it is 2π. For the other exam the circle has radius 3 and so the length is 3π.

5. (16 points) The gravitational force on a 1 kg object at a distance of r meters from the center of the earth is $F = 4 \cdot 10^{14}/r^2$. (F is in Newtons.) The radius of the earth is $6.4 \cdot 10^6$ meters.
(a) Find the work done in moving the object from the surface of the earth to a height 100m above the surface. **Solution:** Let h be the height you lift it. Let r be the distance from the center of the earth. The work to lift it through Δr is $F(r)\Delta r$. So the total work is

$$\int_{6.4 \cdot 10^6}^{6.4 \cdot 10^6 + h} \frac{4 \cdot 10^{14}}{r^2}\, dr = \left[\frac{-4 \cdot 10^{14}}{r}\right]_{6.4 \cdot 10^6}^{6.4 \cdot 10^6 + h} \tag{16}$$

For $h = 100m$ this comes out to $976.5\, N \cdot m$. And for $h = 200m$ it is $1953\, N \cdot m$.
(b) Find the work done in moving the object from the surface of the earth to a point infinitely far from the earth. **Solution:**

$$\int_{6.4 \cdot 10^6}^{\infty} \frac{4 \cdot 10^{14}}{r^2}\, dr = \left[\frac{-4 \cdot 10^{14}}{r}\right]_{6.4 \cdot 10^6}^{\infty} = \frac{4 \cdot 10^{14}}{6.4 \cdot 10^6} = 6.25 \cdot 10^7\, N \cdot m \tag{17}$$

6. (15 points) A dam is in the shape of a semicircle. The top of the dam is straight and 100 ft long. So the distance to the lowest part of the dam is 50 ft. Recall that the pressure in water at a depth of h feet is $62.4h$ in lbs/ft^2. Find the total force on the dam if the surface of the water behind

it is 20 ft. below the top of the dam. **Solution:** Slice it horizontally. Let y be the distance below the top of the dam. So the equation of the dam is $x^2+y^2=50^2$. The area of a slice is $2x\Delta y = 2\sqrt{50^2-y^2}\Delta y$. The slice is at a depth below the surface of $y-20$. So the pressure at the slice is $62.4(y-20)$. So the force on the slice is $62.4(y-20)2\sqrt{50^2-y^2}\Delta y$. So the total force is

$$\int_{20}^{50} 62.4(y-20)2\sqrt{50^2-y^2}\,dy \tag{18}$$

7. (18 points) A pile of gravel is in the shape of a cone. The base of the cone is a circle with radius 20 ft and the height of the cone is 20 ft. The gravel has a density of 100 lbs/ft^3. (The other exam had a pyramid with a radius of 10 ft and a height of 10 ft.)
(a) Find the weight of the gravel pile. **Solution:** Slice it parallel to the ground. Let y be the distance above the ground. The slices are discs. Similar triangles show their radius is $r=20-y$. So the slice volume is $\pi(20-y)^2\Delta y$. Its weight is $100\pi(20-y)^2\Delta y$. So the total weight is

$$\int_0^{20} 100\pi(20-y)^2\,dy = 100\pi\int_0^{20}(400-40y+y^2)\,dy \tag{19}$$

$$= 100\pi\left[400y-20y^2+y^3/3\right]_0^{20} = \frac{800,000\pi}{3} \approx 837,757 lbs \tag{20}$$

For the other problem you get $\frac{100,000\pi}{3} \approx 104,720 lbs$.

(b) Assuming that all the gravel was initially at ground level, how much work did it take to put the gravel in this conical pile? **Solution:** The slice we considered above gets lifted a distance y. So the total work is

$$\int_0^{20} 100\pi(20-y)^2 y\,dy = \frac{4,000,000\pi}{3} \approx 4,188,784 ft-lbs \tag{21}$$

For the other problem we get $\frac{500,000\pi}{6} \approx 261,799 ft-lbs$.