Review: mins, maxs and inflection points  (8/20)

$f(x)$ has a local min at $x = a$ if $f(a) \leq f(x)$ for all $x$ in some neighborhood of $a$.

$f(x)$ over the interval $I$ has a global min at $x = a$ if $f(a) \leq f(x)$ for all $x$ in $I$.

$f(x)$ has a critical point at $a$ if $f'(a) = 0$ (or is undefined).

If $f(x)$ has a local min or max at $a$, then $a$ is a critical point.

Second derivative test: If $f'(a) = 0$ and $f''(a) > 0$, then $f$ has a local min at $x = a$.

First derivative test: If $f'(a) = 0$, $f'(x) < 0$ for $x$ just left of $a$ and $f'(x) > 0$ for $x$ just right of $a$, then $f$ has a local min at $x = a$.

$f(x)$ has an inflection point at $a$ if $f$ changes concavity at $a$. At an inflection point $f''(a) = 0$.

Global min/max: If there is one, it will occur at a local min/max or an endpoint.

Functions with parameters: Everything (number of mins, maxs, inflection points, their locations ...) can depend on the parameter.
Difference between $\int_a^b f(x) \, dx$ and $\int_a^b f(x) \, dx$.

**Fundamental thm of calculus:** If $f(x)$ is continuous on $[a, b]$ and $F'(x) = f(x)$ ($F$ is an antiderivative of $f$), then

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

**Second fundamental thm of calculus:** If $f(x)$ is continuous on $[a, b]$, then

$$\frac{d}{dx} \int_a^x f(u) \, du = f(x)$$

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(u) \, du = f(h(x))h'(x) - f(g(x))g'(x)$$
7.1 Integration by substitution  (8/27)

Chain rule:

\[
\frac{d}{dx} f(u(x)) = f'(u(x)) u'(x)
\]

Guess and check: Guess the antiderivative and check with chain rule. Fudge as needed.

Substitution: (assumes original integral is with respect to \( x \))
1. **Cleverly** choose function \( u(x) \).
2. Compute \( u' = \frac{du}{dx}. \  du = u'dx. \)
3. Express integrand in terms of \( u \).
4. Do the \( u \) integral.
5. Express answer in terms of \( x \).
7.2 Integration by parts  (8/29)

Integration by parts:

\[ \int u v' \, dx = u v - \int u' \, v \, dx \]

\[ \int u \, dv = u v - \int v \, du \]

\[ \int_{a}^{b} u v' \, dx = [u v]_{a}^{b} - \int_{a}^{b} u' \, v \, dx \]

Choosing \( u \) and \( v' \):

1. **Must** be able to compute \( v \) from \( v' \).
2. **Would like** \( u' \) to be simpler than \( u \).
Appendix B: Complex numbers  (8/31)

Basic algebra: \( i = \sqrt{-1}, \quad i^2 = -1 \)
General complex number is \( a + bi \).
Addition/subtraction: obvious
Multiplication: remember \( i^2 = -1 \).
Division:
Complex conjugate of \( z = a + bi \) is \( \bar{z} = a - bi \).

\[
\frac{1}{a + bi} = \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i
\]

Euler’s formula:
\[
e^{i\theta} = \cos(\theta) + i \sin(\theta)
\]
Polar coordinates
\[
re^{i\theta} = r \cos(\theta) + ir \sin(\theta)
\]
Trig functions:
\[
\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}
\]
**Dif Eq 1.1: Simplest dif. eq.  (9/3)**

Definitions: First order differential equation:

\[
\frac{dy}{dx} = g(x, y)
\]

*y* is (unknown) function of *x*.

Initial condition: \(y(x_0) = y_0\)

Special case for chap one:

\[
\frac{dy}{dx} = g(x)
\]

Solving dif eq \(\iff\) integrating \(g(x)\).

**General solution:** \(y(x) = \int g(x)dx + C\)

Initial condition picks out a single solution.
Dif Eq 1.2: Graphical solutions  (9/5)

Strategy: Dif eq gives you derivates of unknown $y(x)$. Use calculus to conclude things.

$$\frac{dy}{dx} = g(x)$$

Sign of $g(x)$ tells you increasing/decreasing.

$$\frac{d^2y}{dx^2} = g'(x)$$

Sign of $g'(x)$ tells you concavity

**Symmetry:**
If $g(x)$ is odd and $y(x)$ is a solution, then $y(-x)$ is a solution.
If $g(x)$ is even and $y(x)$ is a solution, then $-y(-x)$ is a solution.
Dif Eq 1.3: Slope Fields (9/10)

Tangent lines:

\[ y(x + \Delta x) \approx y(x) + \frac{dy}{dx}(x) \Delta x \]

Slope field: If \( y(x) \) solves dif. eq.

\[ \frac{dy}{dx} = g(x, y) \]

then slope at \( (x, y(x)) \) is \( g(x, y(x)) \).

Drawing slope field: Draw a grid of points in \( x, y \) plane. At a point \( (x, y) \) draw a small line segment with slope \( g(x, y) \).

Drawing solution: Start at the initial condition. Draw a curve so that its tangent lines follow the slope field.
**Dif Eq 2.1: Autonomous equations (9/12)**

**Def:** A first order dif. eq. is **autonomous** if it is of the form $\frac{dy}{dx} = g(y)$.

Separation of variables gives implicit equation for $y(x)$.

$$\frac{dy}{g(y)} = dx \quad \Rightarrow \quad \int \frac{dy}{g(y)} = \int dx = x + C$$

Slope field and calculus give you qualitative picture.

Slope field is constant in horizontal direction.
If $y(x)$ is a solution, then $y(x + C)$ is a solution.

An **equilibrium solution** is a solution of the form $y(x) = constant$.
Caution: Separation of variables can “lose” these solutions.

An equilibrium solution is **stable** if solutions that start near the equilibrium solution converge to the equilibrium solution as $x \to \infty$.

An equilibrium solution is **unstable** if solutions that start near the equilibrium solution move away from it as $x \to \infty$. 
**Dif Eq 2.2: Exponential growth/decay**  (9/?)

The simplest model for **growth or decay** is to assume the quantity grows at a rate proportional to the quantity:

\[
\frac{dy}{dt} = ky
\]

The general solution is

\[
y(t) = y_0 e^{kt}
\]

\(y_0\) is the amount at time \(t = 0\).

If \(k > 0\) we have exponential growth.
The **doubling time** \(t_d\) is given by solving \(y(t_d) = 2y_0\).

If \(k < 0\) we have exponential decay.
The **half-life** \(t_h\) is given by solving \(y(t_h) = \frac{1}{2}y_0\).
Calc 7.4: Partial fractions  (9/14)

Algebraic method to integrate rational functions. $P(x)$ and $Q(x)$ are polynomials. Degree of $P$ is lower than that of $Q$. (Long division if needed.)

\[
\frac{P(x)}{Q(x)} = \text{sum of terms}
\]

Factor $Q(x)$ into a product of linear terms and quadratic terms that have no real roots. For each factor:

- **Distinct linear factor** $(x - c)$ Include term
  \[
  \frac{A}{x-c}
  \]

- **Repeated linear factor** $(x - c)^n$ Include terms
  \[
  \frac{A_1}{x-c} + \frac{A_2}{(x-c)^2} + \cdots + \frac{A_n}{(x-c)^n}
  \]

- **Distinct quadratic factor** $q(x)$ Include term
  \[
  \frac{Ax+B}{q(x)}
  \]

- **Repeated quadratic factor** $q^n(x)$ Include terms
  \[
  \frac{A_1 x+B_1}{q(x)} + \frac{A_2 x+B_2}{q^2(x)} + \cdots + \frac{A_n x+B_n}{q^n(x)}
  \]
**Dif Eq. 2.3: Logistic Equation**  
(9/24)

Model for population growth. $t$ is time, $y(t)$ is population

$$\frac{dy}{dt} = ay(b - y)$$

For small $y$, $dy/dt$ proportional to $y$.

Environment can sustain maximum population of $y = b$.

$y = 0$ is **unstable** equilibrium.

$y = b$ is **stable** equilibrium.
Big questions:
Local existence
Global existence
Uniqueness

Existence-Uniqueness Theorem For differential equation

\[
\frac{dy}{dx} = g(x, y)
\]

suppose \(g(x, y)\) and \(\frac{\partial g}{\partial y}\) are defined and continuous in a rectangle which has \((x_0, y_0)\) in its interior. Then there exists a solution through \((x_0, y_0)\) which is defined for \(x\) in an interval with \(x_0\) in its interior. There is no other solution through \((x_0, y_0)\)

Caution: There are \(g(x, y)\) for which solutions can intersect.
**Dif Eq. 2.5: Phase lines (9/28)**

Autonomous differential equation

\[
\frac{dy}{dx} = g(y)
\]

**Phase line:** Line is for \( y \).
Find equilibria (zeroes of \( g(y) \)). Mark them on line.
Find sign of \( g(y) \). Indicate on line with \( \rightarrow\)– for \( g > 0 \), \( \leftarrow\) for \( g < 0 \)
Determine stable, unstable equilibria.

**Derivative test for stable/unstable:**
Let \( c \) be equilibrium solution, so \( g(c) = 0 \).
If \( g'(c) < 0 \) then \( c \) is stable.
If \( g'(c) > 0 \) then \( c \) is unstable.
If \( g'(c) = 0 \) then who knows.

**Monotonicity:**
Non-equilibrium solutions are always increasing or always decreasing.
Dif Eq. 2.6: Bifurcation diagrams (10/1)

If the differential equation contains a parameter, the behavior can change qualitatively as the parameters changes.
In particular, number of equilibrium solutions and their stability can change.
Bifurcation diagram is a plot of all the equilibria as a function of the parameter. Also indicates their stability.
$\textbf{Dif Eq. 3.1: Graphical analysis : }g(x,y) \quad (10/5)$

Now consider the most general first order equation

$$\frac{dy}{dx} = g(x, y)$$

In general, no analytic solution.

**Graphical tools:**
- **Slope fields**
- **Monotonicity:** increasing vs. decreasing : sign of $g(x,y)$
- **Concavity:** sign of $\frac{d^2 y}{dx^2}$, remember $y = y(x)$.
- **Symmetries**
- **Isoclines:** especially $m=0$ isocline:
  Isoclines are curves $g(x, y) = m$. For $m = 0$ they are possible locations of local max or min of solution curves.
**Dif Eq. 3.2: Symmetry, scaling** (10/8)

**Symmetry** for \( \frac{dy}{dx} = g(x, y) \).

If slope field is symmetric about \( y \)-axis, \( g(-x, y) = -g(x, y) \), then the solutions are even functions.

If slope field is symmetric about \( x \)-axis, \( g(x, -y) = -g(x, y) \), and \( y(x) \) is a solution, then \( \overline{y}(x) = -y(x) \) is another solution.

If slope field is symmetric about origin, \( g(-x, -y) = g(x, y) \), and \( y(x) \) is a solution then \( \overline{y}(x) = -y(-x) \) is another solution. In particular the solution through the origin is an odd function.

**Scaling:** If \( y(x) \) solves some differential equation, then for constants \( a \) and \( b \), \( \overline{y}(x) = ay(bx) \) solves a similar equation. You may be able to eliminate some parameters this way.
Calc 7.5, 7.6: Numerical Integration  (10/10)

To compute $\int_{a}^{b} f(x) \, dx$, let $n$ be positive integer.

$x_0 < x_1 < \cdots < x_{n-1} < x_n$ are equally spaced with $a = x_0$, $b = x_n$.

So spacing is $\Delta x = (b - a)/n$.

Riemann Sums:

$LEFT(n) = \sum_{i=1}^{n} f(x_{i-1}) \Delta x, \quad RIGHT(n) = \sum_{i=1}^{n} f(x_i) \Delta x$

Midpoint rule: $MID(n) = \sum_{i=1}^{n} f((x_{i-1} + x_i)/2) \Delta x$,

Trapezoid rule: $TRAP(n) = \sum_{i=1}^{n} \frac{1}{2} [f(x_{i-1}) + f(x_i)] \Delta x$

Simpson’s rule: $SIMPSON(n) = \frac{2}{3} MID(n) + \frac{1}{3} TRAP(n)$

Error of method is $\left| \int_{a}^{b} f(x) \, dx - \text{approximation} \right|$

Order of method: How fast error $\rightarrow 0$ as $N \rightarrow \infty$.

$LEFT, RIGHT$ are first order. Error $\rightarrow 0$ as $1/N$.

$MID, TRAP$ are second order. Error $\rightarrow 0$ as $1/N^2$.

$SIMPSON$ is fourth order. Error $\rightarrow 0$ as $1/N^4$. 

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Over/under estimate:
If \( f \) is increasing on \([a, b]\) then \( \text{LEFT} \leq \int_a^b f(x) \, dx \leq \text{RIGHT} \)
If \( f \) is decreasing on \([a, b]\) then \( \text{RIGHT} \leq \int_a^b f(x) \, dx \leq \text{LEFT} \)
If \( f \) is concave up on \([a, b]\) then \( \text{MID} \leq \int_a^b f(x) \, dx \leq \text{TRAP} \)
If \( f \) is concave down on \([a, b]\) then \( \text{TRAP} \leq \int_a^b f(x) \, dx \leq \text{MID} \)

Extrapolation: If \( \text{error} \approx c/N^p \), then by evaluating approximation at \( N \) and \( 2N \), we can get a better approximation:
\[
\int_a^b f(x) \, dx = \text{APPROX}(N) + \frac{c}{N^p}
\]
\[
\int_a^b f(x) \, dx = \text{APPROX}(2N) + \frac{c}{(2N)^p}
\]
Solve two equations for unknowns \( \int_a^b f(x) \, dx \) and \( c \). Result is a better approximation for \( \int_a^b f(x) \).
Differential equation has the form
\[ \frac{dy}{dx} = f(y)g(x) \]

**Equilibria:** First find all the zeroes (if any) of \( f \). These \( y \) values are equilibrium solutions.

**Solving it:**

\[ \int \frac{dy}{f(y)} = \int g(x) \, dx \]

Do both integrals. Then solve for \( y \) as a function of \( x \).

**Caution:** put in the \(+C\) at the right place.
Want to numerically approximate solution of
\[ \frac{dy}{dx} = g(x, y) \] through \((x_0, y_0)\).

\[ x_n = x_0 + nh. \] Step size is \(h\).

\[ y_n \] is approximation to \(y(x_n)\).
Want to numerically approximate solution of
\[ \frac{dy}{dx} = g(x, y) \] through \((x_0, y_0)\).

\[ x_n = x_0 + nh. \] Step size is \(h\).

\(y_n\) is approximation to \(y(x_n)\).

**Euler:** \( y_n = y_{n-1} + g(x_{n-1}, y_{n-1})h \)
Want to numerically approximate solution of
\[ \frac{dy}{dx} = g(x, y) \text{ through } (x_0, y_0). \]
\[ x_n = x_0 + nh. \text{ Step size is } h. \]
\[ y_n \text{ is approximation to } y(x_n). \]

**Euler:** \[ y_n = y_{n-1} + g(x_{n-1}, y_{n-1})h \]

**Modified Euler or Huen:** \[ y(x_1) = y(x_0) + \frac{1}{2} h (m_0 + k_1) \]
\[ m_0 = g(x_0, y_0) \]
\[ k_1 = g(x_1, y_0 + m_0 h) \]
Want to numerically approximate solution of 
\[ \frac{dy}{dx} = g(x, y) \]  through \((x_0, y_0)\).
\[ x_n = x_0 + nh. \quad \text{Step size is } h. \]
\[ y_n \text{ is approximation to } y(x_n). \]

**Euler:** 
\[ y_n = y_{n-1} + g(x_{n-1}, y_{n-1})h \]

**Modified Euler or Huen:** 
\[ y(x_1) = y(x_0) + \frac{1}{2}h(m_0 + k_1) \]
\[ m_0 = g(x_0, y_0) \]
\[ k_1 = g(x_1, y_0 + m_0h) \]

**Order of methods:**
Error is difference between exact solution and approximation.
It typically goes as \(h^p\).
**Euler:** \(p = 1\)
**Modified Euler (Huen)** \(p = 2\)
**Fourth order Runge-Kutta** \(p = 4\)
**Dif Eq 3.4: Comparison theorem   (10/26)**

**Comparison theorem** Consider the two differential equations
\[
\frac{dy}{dx} = f(x, y) \\
\frac{dy}{dx} = g(x, y)
\]
with the same initial condition \( y(x_0) = y_0 \). Suppose \( f(x, y) < g(x, y) \) for all \( x > x_0 \) and all \( y \). Then the solution to the first differential equation is less than the solution to the second differential equation for \( x > x_0 \) as long as these solutions exist.

**Application** Suppose we have a differential equation \( \frac{dy}{dx} = f(x, y) \) that we can’t solve. Look for a simpler equation \( \frac{dy}{dx} = g(x, y) \) with \( f(x, y) < g(x, y) \) (or \( f(x, y) > g(x, y) \)) that we can solve.
Dif Eq 4.2: Homogeneous coefficients (10/29)

\( g(x, y) \) is homogeneous of degree zero if
\( g(cx, cy) = g(x, y) \) for all \( c \neq 0 \).

For such \( g \) there is a function of one variable, \( G \), such that
\( g(x, y) = G(y/x) \).

Solving \( \frac{dy}{dx} = G(y/x) \).

Try the substitution \( y(x) = xu(x) \).
Should get separable differential equation for \( u \).

Linear solutions:
\( y(x) = mx \) is a solution if \( G(m) = m \).
Dif Eq 4.3: Dif eqs from data  (10/31)

Numerical differentiation: For small $h$,

$$f'(x) \approx \frac{f(x + h) - f(x)}{h}$$

$$f'(x) \approx \frac{f(x + h) - f(x - h)}{2h} \quad \text{Better}$$

Log-log plots: Suppose we have data points $(x_i, y_i)$ and we think $y = cx^p$ but don’t know $p$. Take the $ln$:

$$y = cx^p \quad \Rightarrow \quad ln(y) = ln(c) + p ln(x)$$

So if we plot $ln(y)$ as a function of $ln(x)$ we should see a straight line and the slope is $p$. 
Dif Eq 4.4: Objects in motion  (11/2)

Velocity, acceleration:
Let $t$ be time, $x(t)$ position.
Then velocity is $v(t) = \frac{dx}{dt}$.
Acceleration is $a(t) = \frac{dv}{dt}$.

Newton’s second law:
$F = ma$, $F$ is force
**Dif Eq 5.1: Solving first order linear DE**  (11/7)

Put equation in form $y' + p(x)y = q(x)$

Compute the **integrating factor** $e^{\int p(x)dx} = e^{\int p}$.

Multiply dif. eq. by the integrating factor:

$$e^{\int p} y' + e^{\int p} p(x) y = e^{\int p} q(x)$$

Recognize this as

$$\left[ e^{\int p} y \right]' = e^{\int p} q(x)$$

Integrate:

$$e^{\int p} y = \int e^{\int p} q(x) \, dx$$
Steady states and transients:
Often the solution to a linear differential equation is a sum of two parts.

The part of the solution that goes to zero as $t \to \infty$ is called the transient part.

The remaining part of the solution that does not go to zero as $t \to \infty$ is called the steady state part.

Warning: Steady state is not the same as equilibrium. Equilibrium solutions are constant in time. Steady state solutions can be periodic.
Consider differential equation of the form:

\[ \frac{dy}{dx} + p(x) y = q(x) y^n \]

where \( n \) is any real number besides 1.
Consider differential equation of the form:

\[ \frac{dy}{dx} + p(x) y = q(x) y^n \]

where \( n \) is any real number besides 1.

The substitution \( u = y^{1-n} \) will lead to a linear dif. eq. for \( u \).
**Dif Eq 5.3: Bernoulli’s Equation (11/19)**

Consider differential equation of the form:

\[ \frac{dy}{dx} + p(x) y = q(x) y^n \]

where \( n \) is any real number besides \( 1 \).

The substitution \( u = y^{1-n} \) will lead to a linear dif. eq. for \( u \).

**Note:** If you forget the power of \( y \) for the substitution, just take \( u = y^p \) and your calculations will tell you what \( p \) should be.
Dif Eq 5.3: Bernoulli’s Equation  (11/19)

Consider differential equation of the form:

\[
\frac{dy}{dx} + p(x) y = q(x) y^n
\]

where \( n \) is any real number besides 1.

The substitution \( u = y^{1-n} \) will lead to a linear dif. eq. for \( u \).

Note: If you forget the power of \( y \) for the substitution, just take \( u = y^p \) and your calculations will tell you what \( p \) should be.
Riemann sums and slicing:

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \int_{a}^{b} f(x) \, dx$$
Riemann sums and slicing:

\[
limit_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \int_{a}^{b} f(x) \, dx
\]

1. Slice your problem up. (Introduce coordinates.)
Riemann sums and slicing:

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \int_{a}^{b} f(x) \, dx$$

1. Slice your problem up. (Introduce coordinates.)

2. Write the quantity you want as a sum of the form \( \sum_i f(x_i) \Delta x \)
Riemann sums and slicing:

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \int_{a}^{b} f(x) \, dx
\]

1. Slice your problem up. (Introduce coordinates.)

2. Write the quantity you want as a sum of the form \( \sum_i f(x_i) \Delta x \)

3. Figure out the definite integral this converges to and compute it.
Calc 8.1,8.2: Volumes, Geometry (11/26)

Riemann sums and slicing:

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \int_{a}^{b} f(x) \, dx$$

1. Slice your problem up. (Introduce coordinates.)

2. Write the quantity you want as a sum of the form $\sum_i f(x_i) \Delta x$

3. Figure out the definite integral this converges to and compute it.

Arc length: For graph of $f(x), a \leq x \leq b$ 

$$\text{length} = \int_{a}^{b} \sqrt{1 + f'(x)^2} \, dx$$

For parametric curve $(x(t), y(t)), a \leq t \leq b$

$$\text{length} = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$
Calc 8.4,8.5: Mass, work, pressure  (11/28, 11/30)

Physics:
For constant density, \( mass = density \cdot volume \)
For constant force, \( work = force \cdot distance \)
For constant pressure, \( force = pressure \cdot area \)

If density, force or pressure is not constant, slice the problem up so that the quantity is constant within a slice. Then mass, force or pressure can be computed by an integral.

Units:
In Metric system mass is measured in grams, kilograms,.... Force is measured in Newtons, dynes. A Newton is kilogram-meter/sec\(^2\). Dyne is gram-centimeter/sec\(^2\). \(10^5\) dynes is a Newton. Work is measured in Joules, ergs. A Joule is a Newton - meter. Erg is a dyne-cm.

In English system force is measured in pounds.
1 Newton \(\approx 0.225\) pounds.
Mass is measured in slugs
On Earth, the weight of (gravitational force on) one kilogram is \(9.8\) N \(\approx 2.2\) pounds.