Order

Solve the ODE over some interval $a \leq x \leq b$.

$\text{error} = y(b) - \text{approximation}$

$y(a)$ is initial condition

Typically

$\text{error} \approx ch^p$

Euler $p = 1$ first order
Better methods

\[ y(x_0 + h) = \int_{x_0}^{x_0 + h} y'(x) \, dx + y(x_0) \]

\[ = \int_{x_0}^{x_0 + h} g(x, y) \, dx + y(x_0) \]

\[ = g(x_0, y(x_0)) \, h + y(x_0) \]

**LEFT corresponds to Euler**

What about trapezoid?

\[ \approx \frac{h}{2} \left[ g(x_0, y(x_0)) + g(x_0 + h, y(x_0 + h)) \right] + y(x_0) \]

Approximate \( y(x_0 + h) \approx y(x_0) + g(x_0, y(x_0)) \, h \)

Let \( m_0 = g(x_0, y_0) \)

\( k_1 = g(x_0 + \frac{h}{2}, y_0 + g(x_0, y_0) \, \frac{h}{2}) \)

\[ y(x_0 + h) = y(x_0) + \frac{h}{2} \left( m_0 + k_1 \right) \]

Modified Euler, Huen's method \( p = 2 \)
Even better method

Runge-Kutta fourth order

$p = 4$

Numerical problems

If \( g(x, y) \) is nice then solution curves doesn't don't cross.

Computer doesn't know this.

\[
\frac{dy}{dx} = y^2 \quad y(0) = -1.2
\]

\( h = 0.1 \) \hspace{1cm} Euler

\( y < 0 \) is a solution

\[
y(0.1) = y(0) + y(0, -1.2) \cdot 0.1
\]

\[
= -1.2 + 1(-4) \cdot 0.1
\]

\[
= -2.4
\]
Example

\[ y' = y^2 \]
\[ y(0) = 1 \]

\[ y = \frac{1}{1-x} \]
by sep. of vars

Look at

\[ y' = y^2 + x \]
\[ f(x,y) = y^2 + x \]
\[ g(x,y) = y^2 \]
\[ f(x,y) > g(x,y) \quad \text{for} \quad x > 0 \]

So theorems say solutions to

\[ \frac{dy}{dx} = f(x,y) \]

with initial condition \( y(0) = 1 \)

is above solution to

\[ \frac{dy}{dx} = g(x,y) \quad \text{with same I.C.} \]

So solution to \( \frac{dy}{dx} = y^2 + x \)

must have a vertical asymptote at \( x = a \quad \text{at} \quad a \leq 1 \).