

# § 4.2 Homogeneous Coets

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①

$$\frac{dy}{dx} = g(x, y) \leftarrow \begin{array}{l} \text{can't} \\ \text{solve} \\ \text{in general} \end{array}$$

This section: special class  
of  $g(x, y)$ .

Def  $g(x, y)$  is homogeneous of  
order zero if

$$g(cx, cy) = g(x, y) \quad \forall c \neq 0.$$

For such a  $g$  it can  
be written as

$$g(x, y) = G(y/x)$$

where  $G$  is a function of  
one variable

Examples

$$\textcircled{1} \quad g(x, y) = \frac{xy}{x^2 + y^2}$$

$$g(cx, cy) = \frac{cx \cdot cy}{c^2 x^2 + c^2 y^2} = \frac{xy}{x^2 + y^2}$$

$$\frac{xy}{x^2 + y^2} = \frac{\frac{xy}{x^2}}{\frac{x^2}{x^2} + \frac{y^2}{x^2}} = \frac{y/x}{1 + \left(\frac{y}{x}\right)^2} \quad \text{10/29} \quad \textcircled{2}$$

$$= u\left(\frac{y}{x}\right)$$

$$u(u) = \frac{u}{1 + u^2}$$

$$\textcircled{2} \quad g(x, y) = e^{y/x} \cdot \frac{x+y}{2}$$

$$\textcircled{3} \quad g(x, y) = \frac{\sin x}{\sin y} \leftarrow \text{NOT.}$$


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Solving them

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Trick : let

$$u(x) = \frac{y(x)}{x}$$

i.e.,

$$y(x) = x u(x)$$

Apply sep. of vars to  $u$   
dif. eq.

Example

$$\frac{dy}{dx} = \frac{y}{x+y}$$

$$y = x u$$

$$\frac{dy}{dx} = u + x \frac{du}{dx}$$

$$u + x \frac{du}{dx} = \frac{y}{x+y}$$

$$= \frac{xu}{x+xu} = \frac{u}{1+u}$$

$$4 + x \frac{dy}{dx} = \frac{y}{1+y}$$

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$$x \frac{dy}{dx} = \frac{y}{1+y} - 4$$

$$\frac{dy}{dx} = \frac{1}{x} \left[ \frac{y}{1+y} - 4 \right]$$

$$\frac{du}{\frac{u}{1+u} - 4} = \frac{1}{x} dx$$

~~$$\frac{y}{1+y} - 4 = \frac{y^2 - 4y + 4y - 4}{(1+y)u}$$
$$= \frac{-4}{(1+y)u}$$~~

$$\frac{u - u(1+u)}{1+u} = \frac{-u^2}{1+u}$$

$$- \frac{(1+u) du}{u^2} = \frac{1}{x} dx$$

$$\left(-\frac{1}{u^2} - \frac{1}{u}\right) du = \frac{1}{x} dx \quad \left. \begin{array}{l} 10/29 \\ \textcircled{5} \end{array} \right\}$$

$$\frac{1}{u} - \ln|u| = \ln|x| + C$$

$$e^{\frac{1}{u}} e^{-\ln|u|} = |x| e^C$$

$$e^{\frac{1}{u}} \frac{1}{|u|} = |x| e^C$$

Stuck with implicit solution.

$$\frac{dy}{dx} = G\left(\frac{y}{x}\right)$$

Equilibria ?

If  $G(0) = 0$  then  $y = 0$   
 is a solution

$$G\left(\frac{y}{x}\right) = \frac{y}{x}$$

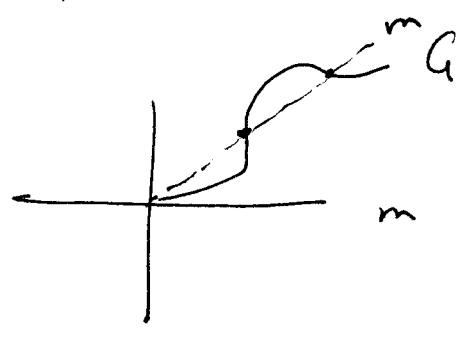
Only possible equilibrium is  $y = 0$ .

If  $y = mx$  a solution?

$$\frac{dy}{dx} = m$$

Need  $m = G\left(\frac{mx}{x}\right) = G(m)$

If  $m$  satisfies  $m = G(m)$ , then  
 $y = mx$  is a solution.



Example

$$\frac{dy}{dx} = \frac{y^2}{x^2}$$

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Equilibrium

$$y = 0$$

Linear?

$$G(u) = u^2$$

$$G(m) = m \Rightarrow m^2 = m$$

$y = x$  is a solution

# § 4.3 modelling

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## word problem

### Newton's law of cooling/heating

My house is at 80.  
Outside is 110.

My AC breaks.

Physicist says that the  
rate of change of temperature  
is proportional to ~~the~~ the  
temperature difference between  
inside and outside.

Solution :  $t = \text{time}$   
 $T(t) = \text{temperature}$

rate =  
derivative

$$\frac{dT}{dt} \propto \text{temp dif.}$$

$\propto \leftarrow$  proportional to

$$\text{temp dif.} = T(t) - 110$$

$$\frac{dT}{dt} = k [T(t) - 110], \quad k < 0$$

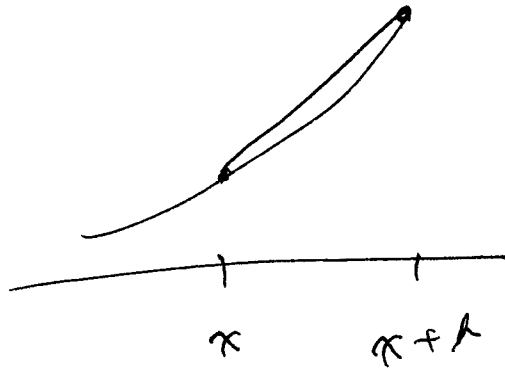
$$\frac{dT}{dt} = k(110 - T(t)), \quad k > 0$$

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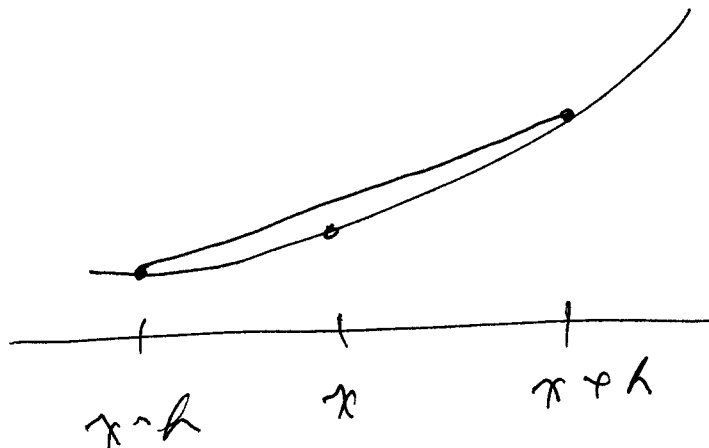
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Numerical      derivatives

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$



$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h}$$



↑  
Better