1. (14 points total) Find the following

\[\int \theta \cos \theta \, d\theta\]

by parts

\[\begin{align*}
U &= \theta, & V' &= \cos \theta \\
U' &= 1, & V &= \sin \theta
\end{align*}\]

\[\int \theta \cos \theta \, d\theta = \theta \sin \theta - \int \sin \theta \, d\theta\]

\[= \theta \sin \theta + \cos \theta + C\]

\[\frac{d}{dx} \int_{-2}^{\sqrt{x}} t \sinh(t^2) \, dt\]

Second fundamental theorem of calculus

\[= \sqrt{x} \sinh \left( \sqrt{x}^2 \right) \frac{d}{dx} \sqrt{x}\]

\[= \sqrt{x} \sinh (1x1) \frac{d}{dx} 1x1\]

\[= \frac{1}{2} \sinh (1x1)\]
2. (21 points) Find the following

\[ \int \frac{e^x}{(2 + e^x)^3} \, dx \]

Sub \( u = 2 + e^x \)
\[ u' = e^x \]
\[ \int \frac{du}{u^3} = -\frac{1}{2} u^{-2} \]
\[ = -\frac{1}{2} (2 + e^x)^{-2} + C \]

\[ \int_0^{\pi/2} \sin \theta \cos^6 \theta \, d\theta \]
Sub \( u = \cos \theta \)
\[ u' = -\sin \theta \]
\[ -\int u^6 \, du = -\frac{1}{7} u^7 + C \]
\[ = -\frac{1}{7} [\cos^7(\pi/2) - \cos^7(0)] = \frac{1}{7} \]

\[ \int \frac{y^3}{\sqrt{y^2 + 1}} \, dy \]
Sub \( u = y^2 + 1 \)
\[ u' = 2y \, dy \]
\[ = \int \frac{y^2 \, du}{\sqrt{u}} = \int \frac{(u^{-1})^{1/2} \, du}{\sqrt{u}} \]
\[ = \frac{1}{2} \int (\sqrt{u} - u^{-1/2}) \, du = \frac{1}{2} \left( \frac{2}{3} u^{3/2} - 2 u^{1/2} \right) + C \]
\[ = \frac{1}{3} (y^2 + 1)^{3/2} - (y^2 + 1)^{1/2} + C \]
3. (22 points) The slope field for the differential equation \( \frac{dy}{dx} = g(x) \) is:

(a) Sketch the two solutions that satisfy \( y(0) = 1 \) and \( y(0) = -1 \) on the slope field.

(b) Suppose \( y(x) \) is a solution and \( c \) is an arbitrary constant. Which of the following is always a solution of the differential equation?

- \( y(x + c) \) **Not always a solution**
- \( y(x) + c \) **Always a solution**
- \( y(-x) \) **Not always a solution**
- \( -y(x) \) **Always a solution**

(c) Sketch the graph of \( g(x) \) on the axes below.
4. (12 points) Find the solution that passes through \((0, -1)\) for the differential equation \(\frac{dy}{dx} = x \sin(x^2)\)

Substitute \(u = x^2\):

\[
\int x \sin(x^2) \, dx = \frac{1}{2} \int \sin(u) \, du
\]

\[
= \frac{1}{2} \cos(x^2) + C
\]

So \(y(x) = -\frac{1}{2} \cos(x^2) + C\)

\(y(0) = -1\) \(\Rightarrow\) \(-1 = -\frac{1}{2} \cos(0) + C\)

\(\Rightarrow\) \(C = -\frac{1}{2}\)

So \(y(0) = -\frac{1}{2} \cos(x^2) - \frac{1}{2}\)

5. (15 points) For the differential equation below, determine where the solutions are increasing, decreasing, concave up, concave down.

\[
\frac{dx}{dt} = \frac{t}{t^2 + 1}
\]

\(\frac{t}{t^2 + 1} > 0\) if \(t > 0\) so it is increasing here.

\(\frac{t}{t^2 + 1} < 0\) if \(t < 0\) so it is decreasing there.

\[
\frac{d^2x}{dt^2} = \frac{(t^2 + 1) - t(2t)}{(t^2 + 1)^2} = \frac{1 - t^2}{(t^2 + 1)^2}
\]

\((t^2 + 1)^2\) is always \(> 0\).

If \(-1 < t < 1\) then \((1 - t^2) > 0\), so concave up there.

If \(t > 1\) or \(t < -1\), \(1 - t^2 < 0\), so concave down there.
6. (16 points) Define the function $E(x)$ by

$$E(x) = \int_0^x e^{-t^2} \, dt$$

It satisfies the differential equation

$$\frac{dy}{dx} = e^{-x^2}$$

(a) Given that $E(1) = 0.746824 \cdots$, find $y(1)$ where $y(x)$ is the solution of the above differential equation that satisfies $y(0) = 2$.

General solution: $E(x) + c = y(x)$

$E(0) = 0$. So $c = 2$. Need $0 + c = 2$, so $y(x) = E(x) + 2$.

So $y(1) = E(1) + 2 = 2.746824$

(b) Now suppose we want to find the solution of the differential equation and initial condition

$$\frac{dy}{dx} = xe^{-x^4}, \quad y(0) = 0$$

Find the value of this solution when $x = 1$.

$y(1) = \int_0^1 u e^{-u^4} \, du$

So $y(1) = \int_0^1 u e^{-u^4} \, du = \frac{1}{4} E(1) = \frac{1}{4}(0.746824)$

(c) For the differential equation and initial condition given in part (b), express the solution in terms of the function $E(x)$.

$y(x) = \int_0^x u e^{-u^4} \, du$. Sub $u = u^2$

$= \int_0^{x^2} \frac{1}{2} e^{-u^2} \, du = \frac{1}{2} E(x^2)$

Can check using chain rule

$$\frac{d}{dx} \left( \frac{1}{2} E(x^2) \right) = \frac{1}{2} e^{\left(-x^2\right)} 2x = x e^{-x^2}$$